The Dirichlet Problem for the Biharmonic Equation in a C^1 Domain in the Plane

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Introduction

In this paper we solve the Dirichlet problem for the biharmonic equation $\Delta^2 u = 0$ on a bounded simply connected C^1 domain $\Omega \subset \mathbf{R}^2$ with boundary data in the space $L_1^p(\partial\Omega) \times L^p(\partial\Omega)$, $p > 1$. More precisely, we show that the solution can be represented as a multiple layer potential of the type introduced by Agmon [1].

The multiple layer potential for constant coefficient elliptic equations (of arbitrary order) in two variables with no lower order terms was introduced by Agmon [1]. Agmon used the multiple layer potential to solve the Dirichlet problem for elliptic equations of order $2m$ on domains contained in $\mathbf{R}^2$ with $C_{1+\beta}$ boundaries and $C_{m-1,\alpha}$ boundary data where $\alpha > 0$ and $\beta > 1/2$. In this paper we extend Agmon's result for the biharmonic operator to C^1 domains with L^p-type boundary data.

The Dirichlet problem for Laplace's equation in a C^1 domain in $\mathbf{R}^n$ was solved by Fabes, Jodeit, and Rivière [4]. Their solution is in the form of a classical double layer potential in $\mathbf{R}^n$. While our proof follows the same general outline as [4] there are several significant differences. First of all, unlike Laplace's equation, Green's formula depends on the particular coordinate system. Thus in analyzing the multiple layer potential near the boundary one has to use a representation of it in the appropriate coordinate system. Secondly, in the case of Laplace's equation, crucial cancellation properties of the boundary kernel of the double layer potential needed to establish compactness of boundary integral operators were apparent from the simplicity of the kernels. In the biharmonic equation these cancellation properties are a consequence of special properties of the conjugate Neuman operators. These operators arise in the boundary integrals of Green's formula which in turn comes from what Agmon refers to as the "distinguished bilinear form." Thirdly, in Laplace's equation the boundary integral of the double layer potential maps $L^p(\partial\Omega)$ boundedly into itself, $1 < p < \infty$, and hence its adjoint can be realized as a bounded integral operator on $L^q(\partial\Omega)$, $1/p + 1/q = 1$. The adjoint is in fact realized as the normal derivative of the single layer potential at the boundary. In our case, the adjoint acts on the dual space of $L_1^p(\partial\Omega) \times L^p(\partial\Omega)$ which involves distributions. We are able to realize the adjoint operator

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as certain differential operators acting on a "lower order potential" at the boundary in the sense of distributions. This realization presents certain technical difficulties. Finally we note that as in the Fabes, Jodeit, and Rivière paper [4] on Laplace's equation the crucial estimates in the biharmonic equation are applications of the celebrated theorem of A. P. Calderón on the Cauchy integral along Lipschitz curves [2].

We note that this problem was originally posed by N. M. Rivère (see dedication to the proceedings of the Williamstown Conference on Harmonic Analysis [5]). We wish to thank Gene Fabes for bringing this problem to the attention of the conference and for several helpful suggestions. We also wish to thank Paul Wenston and Gary Roberts for acting as a sounding board and for offering many useful suggestions.

1. Summary

In the remaining sections we introduce notation and prove various results which will ultimately prove the following:

Theorem. *Let Ω be a simply connected bounded C^1 domain in $\mathbf{R}^2$. Let $f \in L_1^p(\partial\Omega)$, $g \in L^p(\partial\Omega)$. Then there exists a function u with $\Delta^2 u = 0$ in Ω, $u|_{\partial\Omega} = f$, and $\partial_n u|_{\partial\Omega} = g$. More precisely, u and its inner normal derivative $\partial_n u$ have non-tangential limits* a.e. *on $\partial\Omega$ (with respect to arclength measure) equal to f and g respectively.*

The proof consists of three basic steps. For $\dot{f} \in \mathcal{B}_p$, a set of compatible triples of boundary data, (see Section 2.4), we analyze an operator of the form

$$\mathcal{L}\dot{f}(P) = \text{p.v.} \int_{\partial\Omega} \dot{f}(Q)\mathcal{L}(P,Q)\,ds(Q)$$

where $\mathcal{L}(P,Q)$ is a square matrix of kernels. These kernels are various differential operators applied to a fundamental solution for the biharmonic operator. We show that $\mathcal{L}\dot{f}$ exists almost everywhere and gives rise to a compact operator from $\mathcal{B}_p$ into $L_1^p(\partial\Omega) \times L^p(\partial\Omega) \times L^p(\partial\Omega)$. The second step is to introduce and analyze the multiple layer potential $u = u(\dot{f};X)$, $\dot{f} \in \mathcal{B}_p$, $X \in \Omega$. Letting $\dot{u}(X) = (u(X), u_x(X), u_y(X))$, we then show that $\dot{u}(X)$ has non-tangential limits almost everywhere as $X \to \partial\Omega$. These boundary values can be written in the form $(I + \mathcal{K})\dot{f}$ where $\mathcal{K}$ is closely related to $\mathcal{L}$. In particular $\mathcal{K}$ is compact from $\mathcal{B}_p$ to itself. The last step is to appeal to the Fredholm Theory and establish that $I + \mathcal{K}$ maps $\mathcal{B}_p$ onto itself by showing that $I + \mathcal{K}$ is invertible. More precisely we shall prove that the adjoint of $I + \mathcal{K}$ is invertible. Once this is established the solution is given by the multiple layer potential of $(I + \mathcal{K})^{-1}\dot{f}$.

The remainder of this paper is divided into four chapters. In Chapter 2 we introduce some notation, discuss the distinguished bilinear form and the corresponding Green's formula, discuss the space of boundary data, introduce the fundamental solution, and describe the multiple layer potential. In Chapter 3 we analyze the boundary integrals and show that they give rise to a compact operator $\mathcal{H}$ on the space of boundary data. In Chapter 4 we show that the multiple layer potential and its derivatives have boundary values a.e. via non-tangential maximal estimates. We also identify these boundary values as $(I + \mathcal{H})\dot{f}$. In Chapter 5 we discuss the dual space of our space of boundary data, and introduce the lower order potential with density in the dual of $L_1^p \times L^p \times L^p$. Finally, we use the lower order potential with density $\dot{\theta}$ in the kernel of $(I + \mathcal{H})^*$ to show that $(I + \mathcal{H})^*$ is one-to-one and hence that $I + \mathcal{H}$ is invertible. This suffices to prove the main result of this paper.

2. Preliminaries

2.1. Notation. Throughout this paper Ω will denote a bounded simply connected C^1 domain in $\mathbf{R}^2$. Points in Ω will generally be denoted by X and Y while points on $\partial\Omega$ will generally be denoted by P and Q. We let ds stand for arclength measure on $\partial\Omega$ and we assume that as one traverses $\partial\Omega$ in the direction of increasing arclength, the domain Ω is on the left. We let $\vec{t}_p$ denote the unit tangent vector at $P \in \partial\Omega$ in the direction of increasing arclength while $\vec{n}_p$ denotes the unit inner normal vector at P.

In general $\dot{f}$ will denote a row vector of functions (f,g,h) and $\dot{f}^T$ will denote the corresponding column vector—the transpose of $\dot{f}$. Matrices of functions or differential operators will be denoted by $\mathcal{A}$, $\mathcal{B}$, ... and if $\mathcal{A}$ is a matrix $\mathcal{A}^T$ will denote its transpose. Whenever we encounter the product of a function and a differential operator, if the function is to the left of the operator, the operator is simply multiplied by the function, whereas if the function is to the right of the differential operator, the differential operator acts on the function.

We also recall the definition of a C^1 domain. A domain $\Omega \subset \mathbf{R}^2$ is a C^1 domain if for each point $P \in \partial\Omega$, there exists a ball $B(P,\delta)$ of positive radius δ centered at P, and a coordinate system (x,y) of $\mathbf{R}^2$ with origin at P such that with respect to this coordinate system, $\Omega \cap B(P,\delta) = \{(x,y): x \in \mathbf{R}, y > \phi(x)\} \cap B(P,\delta)$ where $\phi \in C_0^1(\mathbf{R})$ and $\phi'(0) = \phi(0) = 0$. We recall that if Ω is a C^1 domain and $\varepsilon > 0$ is given, we can find a finite covering of $\partial\Omega$ by balls $B(P_j,\delta_j)$, $j = 1, \ldots, n$, such that $\Omega \cap B(p_j,\delta_j) = \{(x,y): y > \phi_j(x)\} \cap B(P_j,\delta_j)$ with $\phi_j \in C_0^1(\mathbf{R})$ and $\|\phi_j'\|_\infty \leq \varepsilon$. Throughout this paper there will be several different places where we need to assume $\|\phi'\|_\infty$ is sufficiently small, but since in the end there will be only finitely many such conditions imposed, we may assume our covering is picked so that all the assumptions are satisfied.

2.2. The distinguished bilinear form and Green's formula. Throughout this section (x,y) will denote global coordinates of $\mathbf{R}^2$ and u and v will denote functions which are C^∞ in a neighborhood of $\bar{\Omega}$.

For $\Delta^2 = (\partial_x^2 + \partial_y^2)^2$ we have the factorization

$$(\partial_y - i\partial_x)^2(\partial_y + i\partial_x)^2 = M(\partial_x,\partial_y)\bar{M}(\partial_x,\partial_y).$$

Letting $M(u) = M(\partial_x,\partial_y)u = u_{yy} - 2iu_{yx} - u_{xx}$ and $\bar{M}(v) = \bar{M}(\partial_x,\partial_y)v = v_{yy} +$

$2i\, v_{yx} - v_{xx}$ we define what Agmon calls the distinguished bilinear form

$$B[u,v] = \mathrm{Re} \int\int_{\Omega} M(u)\bar{M}(v)\,dxdy \tag{2.2.1}$$
$$= \int\int_{\Omega} (u_{yy} - u_{xx})(v_{yy} - v_{xx}) + 4\,u_{yx} v_{yx} dxdy.$$

To obtain our Green's formula we integrate $B[u,v]$ by parts first transferring the x derivatives of u onto v and then transferring the y derivatives. We obtain

$$B[u,v] = \int\int_{\Omega} u\Delta^2 v\, dxdy$$
$$+ \int_{\partial\Omega} u(-v_{xxx} + v_{yyx}) + u_x(v_{xx} - v_{yy}) + 4\, u_y v_{yx} dy \tag{2.2.2}$$
$$+ \int_{\partial\Omega} u(v_{yyy} + 3\, v_{xxy}) + u_y(v_{xx} - v_{yy})\,dx$$
$$\equiv \int\int_{\Omega} u\Delta^2 v\, dxdy + \int_{\partial\Omega} \Lambda_1[u,v]\,dy + \int_{\partial\Omega} \Lambda_2[u,v]\,dx.$$

Let $\dot{u} = (u,u_x,u_y)$, $A(v) = (A_1(v),A_2(v),A_3(v))$, and $C(v) = (C_1(v),C_2(v),C_3(v))$ where

$$A_1(v) = A_1(\partial_x,\partial_y)(v) = (-\partial_{xxx} + \partial_{xyy})v$$
$$A_2(v) = A_2(\partial_x,\partial_y)(v) = (\partial_{xx} - \partial_{yy})v \tag{2.2.3}$$
$$A_3(v) = A_3(\partial_x,\partial_y)(v) = (4\,\partial_{xy})v$$

and

$$C_1(v) = C_1(\partial_x,\partial_y)(v) = (\partial_{yyy} + 3\,\partial_{xxy})v$$
$$C_2(v) = C_2(\partial_x,\partial_y)(v) = 0$$
$$C_3(v) = C_3(\partial_x,\partial_y)(v) = (\partial_{xx} - \partial_{yy})v.$$

We then have $\Lambda_1[u,v] = \dot{u}A(v)^T$ and $\Lambda_2[u,v] = \dot{u}C(v)^T$. Let

$$K(v) = K(\partial_x,\partial_y)(v) = \frac{dy}{ds}A(v) + \frac{dx}{ds}C(v).$$

Then our Green's formula (2.2.2) becomes

$$B[u,v] = \int\int_{\Omega} u\Delta^2 v dxdy + \int_{\partial\Omega} \dot{u}K(v)^T ds. \tag{2.2.4}$$

We now analyze our Green's formula under changes of variable. Suppose $(x',y') = (x,y)\mathfrak{R}^T$ are new coordinates of $\mathbf{R}^2$ where

$$\mathfrak{R} = \begin{pmatrix} n_1 & n_2 \\ -n_2 & n_1 \end{pmatrix},$$

$n_1^2 + n_2^2 = 1$, is a rotation. Then $\partial_x = n_1 \partial_{x'} - n_2 \partial_{y'}$ and $\partial_y = n_2 \partial_{x'} + n_1 \partial_{y'}$. Then

$$\begin{aligned} M(\partial_x,\partial_y) &= M(n_1 \partial_{x'} - n_2 \partial_{y'}, n_2 \partial_{x'} + n_1 \partial_{y'}) \\ &= [(n_2 \partial_{x'} + n_1 \partial_{y'}) - i(n_1 \partial_{x'} - n_2 \partial_{y'})]^2 \\ &= [(n_1 + i n_2)(\partial_{y'} - i \partial_{x'})]^2 = (n_1 + i n_2)^2 M(\partial_{x'},\partial_{y'}). \end{aligned} \tag{2.2.5}$$

Similarly one has $\bar{M}(\partial_x,\partial_y) = (n_1 - in_2)^2 \bar{M}(\partial_{x'},\partial_{y'})$. Thus

$$M(\partial_x,\partial_y)\bar{M}(\partial_x,\partial_y) = M(\partial_{x'},\partial_{y'})\bar{M}(\partial_{x'},\partial_{y'})$$

since $n_1^2 + n_2^2 = 1$. Thus if we integrate by parts in our new variables and let $\dot{u}' = (u,u_{x'},u_{y'})$ we obtain

$$B[u,v] = \int\int_{\Omega} u\Delta^2 v \, dx' \, dy' + \int_{\partial\Omega} \dot{u}' K(\partial_{x'},\partial_{y'})(v)^T ds. \tag{2.2.6}$$

The most important feature of (2.2.6) is that the explicit form of the differential operators in K is unchanged.

We conclude this section by noting that $B[u,v]$ is referred to as the "distinguished" bilinear form because of the operators $C_1(\partial_x,\partial_y)$ and $C_3(\partial_x,\partial_y)$. One can show that these operators are essentially what Agmon calls the conjugate Neumann operators. These operators occur very naturally in the solution of the Dirichlet problem for the half-plane. For more details the reader is referred to Agmon [1], pages 191, 192, and 195.

2.3. The fundamental solution. We will use the same fundamental solution used by Agmon in [1], page 189. The fundamental solution with pole $X = (x,y)$ and variable point $Q = (s,t)$ is given by

$$F(X - Q) = \mathrm{Re}\left\{ \frac{1}{\pi^2} \int_{\gamma} \frac{[(x - s) + (y - t)\zeta]^2 \log[(x - s) + (y - t)\zeta]}{2!(\zeta^2 + 1)^2} d\zeta \right\} \tag{2.3.1}$$

where γ is a rectifiable simple closed curve in the upper half-plane containing i. For $u \in C_0^4(\mathbf{R}^2)$ one has

$$u(X) = -\frac{1}{2} \int\int_{\mathbf{R}^2} F(X - Q)\Delta^2 u(Q) dA(Q) \tag{2.3.2}$$

where $dA(Q)$ denotes standard Lebesgue measure in $\mathbf{R}^2$. Using Cauchy's formula, one can show that

(2.3.3) $F(X-Q)$

$$= \frac{1}{4\pi}[(x-s)^2+(y-t)^2]\log[(x-s)^2+(y-t)^2]^{1/2}+\frac{1}{4\pi}(y-t)^2$$

or if $\vec{n}$ denotes a unit vector in the positive y direction one has

$$F(X-Q)=\frac{1}{4\pi}\{|X-Q|^2\log|X-Q|+\langle X-Q,\vec{n}\rangle^2\}. \tag{2.3.4}$$

From (2.3.4) it is clear that if (x',y') is a new set of coordinates obtained by applying a rotation to the coordinates (x,y) and $\tilde{F}(X-Q)$ represents the contour integral in (2.3.1) in the new coordinates (x',y'), then $\tilde{F}(X-Q)$ and $F(X-Q)$ differ by a second degree polynomial. In particular, when applying differential operators of degree ≥ 3 to $F(X-Q)$ we may choose a representation for $F(X-Q)$ in any coordinate system. Finally, if P_0 is a point on $\partial\Omega$ and we choose coordinates parallel to the vectors $\vec{t}_{P_0}$ and $\vec{n}_{P_0}$ respectively, we will denote the integral in (2.3.1) by $F(X-Q;\vec{n}_{P_0})$. For future reference we have

(2.3.5) $F(P-Q;\vec{n}_{P_0})$

$$= \mathrm{Re}\left\{\frac{1}{\pi^2}\int_\gamma \frac{\langle P-Q,\vec{n}_{P_0}\zeta+\vec{t}_{P_0}\rangle^2\log\langle P-Q,\vec{n}_{P_0}\zeta+\vec{t}_{P_0}\rangle\,d\zeta}{(\zeta^2+1)^2}\right\}$$

$$= \frac{1}{4\pi}[\langle P-Q,\vec{t}_{P_0}\rangle^2+\langle P-Q,\vec{n}_{P_0}\rangle^2]\log[\langle P-Q,\vec{t}_{P_0}\rangle^2+\langle P-Q,\vec{n}_{P_0}\rangle^2]^{1/2}$$

$$+\frac{1}{4\pi}\langle P-Q,\vec{n}_{P_0}\rangle^2.$$

2.4. Boundary data. As in [4] we say a function f defined on $\partial\Omega$ belongs to the space $L_1^p(\partial\Omega)$ provided that for every finite covering $\{B(P_j,\delta_j)\}_{j=1}^n$ of $\partial\Omega$ as in the definition of a C^1 domain and for any $\psi\in C_0^\infty(\mathbf{R}^2)$ with support in a single $B(P_j,\delta_j)$ the function $\tilde{\psi f}\equiv\psi(x,\phi(x))f(x,\phi(x))$ has a distributional derivative in $L^p(\mathbf{R})$. It is not difficult to see that $\tilde{\psi f}$ must agree a.e. with an absolutely continuous function and hence we may assume $\tilde{\psi f}$ has a pointwise derivative a.e. This in turn easily implies f has a derivative with respect to arclength almost everywhere and furthermore this derivative is in $L^p(\partial\Omega;ds)$.

Let $\dot{f}=(f,g,h)\in L_1^p(\partial\Omega)\times L^p(\partial\Omega)\times L^p(\partial\Omega)$, $1<p<\infty$. We say $\dot{f}$ is a *compatible triple* if

$$\frac{df}{ds}=g\frac{dx}{ds}+h\frac{dy}{ds}\qquad \text{a.e. } (ds). \tag{2.4.1}$$

We let $\mathcal{B}_p$ denote the closed subspace of $L_1^p(\partial\Omega)\times L^p(\partial\Omega)\times L^p(\partial\Omega)$ consisting of compatible triples. We will actually prove that for $\dot{f}\in\mathcal{B}_p$ there exists a function u with $\Delta^2u=0$ in Ω and $\dot{u}(X)\equiv(u(X),u_x(X),u_y(X))\to\dot{f}(P)$ as $X\to P$ nontangentially for a.e. $P\in\partial\Omega$. We note that this implies that we can solve the

Dirichlet problem as stated earlier. For suppose $F \in L_1^p(\partial D)$ and $G \in L^p(\partial D)$ are the boundary data for u and its normal derivative. If $\vec{i}$ and $\vec{j}$ are unit vectors parallel to the x and y axes respectively, given $P \in \partial\Omega$, we let $\vec{t}_p = t_1(P)\vec{i} + t_2(P)\vec{j}$ and $\vec{n}_p = -t_2(P)\vec{i} + t_1(P)\vec{j}$. We then define

$$\dot{f} = (f,g,h) = \left(F,t_1(P)\frac{dF}{ds} - t_2(P)G,t_2(P)\frac{dF}{ds} + t_1(P)G\right).$$

Then $\dot{f}$ is a compatible triple in $\mathcal{B}_p$ and the solution u with $\dot{u} = (u,u_x,u_y) \to \dot{f}$ also satisfies $u(X) \to F(P)$ and $\nabla u(X) \cdot \vec{n}_P \to G(P)$ a.e. (P) as $X \to P$ non-tangentially. We note that the idea of working with compatible triples was introduced by Agmon. The main reason for working with compatible triples is the dependence of the distinguished Green's formula on a particular coordinate system.

In various places throughout this paper we will need to work with a dense class of $\mathcal{B}_p$. We note that the subspace

$$\dot{C}_1 = \{\dot{f} = (f,g,h) : \dot{f} \in C^1(\partial\Omega) \times C(\partial\Omega) \times C(\partial\Omega) \text{ and } \dot{f} \text{ is compatible}\}$$

is dense in $\mathcal{B}_p$. To see this suppose $\dot{f} = (f,g,h)$ is a compatible triple in $\mathcal{B}_p$. Set $F = f$ and $G = -t_2(P)g + t_1(P)h$. Then $F \in L_1^p(\partial\Omega)$ and $G \in L^p(\partial\Omega)$ and so there exists sequences F_n and G_n in $C^1(\partial\Omega)$ and $C(\partial\Omega)$ respectively with $F_n \to F$ in $L_1^p(\partial\Omega)$ and $G_n \to G$ in $L^p(\partial\Omega)$. Then if

$$\dot{f}_n = \left(F_n,t_1(P)\frac{dF_n}{ds} - t_2(P)G_n,t_2(P)\frac{dF_n}{ds} + t_1(P)G_n\right)$$

it is easy to see that each $\dot{f}_n$ is compatible with $\dot{f}_n \to \dot{f}$ in $\mathcal{B}_p$. It is worth noting that by a version of the Whitney extension theorem (see page 200 of [1]) $\dot{C}_1$ coincides with the restrictions to $\partial\Omega$ of functions which are C^1 in $\mathbf{R}^2$. Thus $\{(f,f_x,f_y)|_{\partial\Omega} : f \in C^\infty(\mathbf{R}^2)\}$ is dense in $\mathcal{B}_p(\partial\Omega)$.

We conclude this section with a brief discussion of $\mathcal{B}_p^*$, the dual space of $\mathcal{B}_p$. If we define two norms on $\mathcal{B}_p$ by

$$\|\dot{f}\|_{p,1} = \|f\|_{L_1^p} + \|g\|_{L^p} + \|h\|_{L^p}$$

and

$$\|\dot{f}\|_{p,0} = \|f\|_{L^p} + \|g\|_{L^p} + \|h\|_{L^p},$$

then it is easy to see that because of compatibility, these norms are equivalent. Thus if we consider $\mathcal{B}_p$ as a closed subspace of $L^p(\partial\Omega) \times L^p(\partial\Omega) \times L^p(\partial\Omega)$, we may identify $\mathcal{B}_p^*$ as the quotient space $(L^q(\partial\Omega) \times L^q(\partial\Omega) \times L^q(\partial\Omega))/\mathcal{B}_p^\perp$ where $1/p + 1/q = 1$ and $\mathcal{B}_p^\perp$ is the annihilator of $\mathcal{B}_p$. Let $\dot{\theta} = (\theta,\phi,\psi)$ denote a triple of functions in $L^q(\partial\Omega)$. We denote the action of $\dot{\theta}$ on $\dot{f} \in \mathcal{B}_p$ by

$$\text{(2.4.2)} \qquad \langle \dot{f},\dot{\theta}\rangle = \int_{\partial\Omega} f(Q)\theta(Q) + g(Q)\phi(Q) + h(Q)\psi(Q)ds(Q).$$

It is interesting to note that linear functionals of the form

$$\Psi(\dot{f}) = \int_{\partial\Omega} \frac{df}{ds}(Q)\psi(Q)\,ds(Q)$$

with $\psi \in L^q(\partial\Omega)$ correspond to triples of the form $(0,\psi(dx/ds),\psi(dy/ds))$ in the above representation of $\mathcal{B}_p^*$. We now describe $\mathcal{B}_p^\perp$. Suppose $\dot{\theta} \in \mathcal{B}_p^\perp$. Since $\langle(1,0,0),\dot{\theta}\rangle = 0$ it follows that $\int_{\partial\Omega}\theta(Q)\,ds(Q) = 0$ and consequently there exists $\tilde{\theta}(s) \in L^q(\partial\Omega)$ with $d\tilde{\theta}/ds = \theta$ almost everywhere. Clearly any triple of the form $(\theta,\tilde{\theta}(dx/ds),\tilde{\theta}(dy/ds))$ is in $\mathcal{B}_p^\perp$. On the other hand, since

$$\int_{\partial\Omega} \theta(Q)f(Q) + \phi(Q)g(Q) + \psi(Q)h(Q)\,ds(Q)$$

$$= \int_{\partial\Omega}\left(\phi(Q) - \tilde{\theta}\frac{dx}{ds}(Q)\right)g(Q) + \left(\psi(Q) - \tilde{\theta}\frac{dy}{ds}(Q)\right)h(Q)\,ds(Q),$$

it is not difficult to see that $\dot{\theta} \in \mathcal{B}_p^\perp$ implies $\dot{\theta}$ is of the form $(\theta,\tilde{\theta}(dx/ds),\tilde{\theta}(dy/ds))$.

2.5. The multiple layer potential. As in Chapter 6 of Agmon [1], for a compatible triple $\dot{f} \in \mathcal{B}_p$ and $X \notin \partial\Omega$ we define the multiple layer potential $u(X)$ or $u(\dot{f};X)$ by

$$u(X) = \int_{\partial\Omega} \dot{f}(Q)K(\partial_x^Q,\partial_y^Q)^T F(X-Q)\,ds(Q) \tag{2.5.1}$$

where

$$(2.5.2)\quad K(\partial_x^Q,\partial_y^Q)^T F(X-Q)$$

$$= \frac{dy}{ds}(Q)\begin{pmatrix} A_1(\partial_x^Q,\partial_y^Q)F(X-Q) \\ A_2(\partial_x^Q,\partial_y^Q)F(X-Q) \\ A_3(\partial_x^Q,\partial_y^Q)F(X-Q)\end{pmatrix} + \frac{dx}{ds}(Q)\begin{pmatrix} C_1(\partial_x^Q,\partial_y^Q)F(X-Q) \\ C_2(\partial_x^Q,\partial_y^Q)F(X-Q) \\ C_3(\partial_x^Q,\partial_y^Q)F(X-Q)\end{pmatrix}$$

and the operators $A_i(\partial_x^Q,\partial_y^Q)$, $C_i(\partial_x^Q,\partial_y^Q)$, $1 \le i \le 3$, are defined in (2.2.3). $F(X-Q)$ is the fundamental solution as given by (2.3.1) and the superscript Q on the differential operators indicates the variable on which they act. In addition to $u(X)$ we consider $\dot{u}(X) = (u(X),u_x(X),u_y(X))$. For $X \notin \partial\Omega$ we may compute $u_x(X)$ and $u_y(X)$ by simply differentiating under the integral sign. We have

$$u_x(X) = \int_{\partial\Omega} \dot{f}(Q)\,\partial_x^X K(\partial_x^Q,\partial_y^Q)^T F(X-Q)\,ds(Q)$$

and

$$u_y(X) = \int_{\partial\Omega} \dot{f}(Q)\,\partial_y^X K(\partial_x^Q,\partial_y^Q)^T F(X-Q)\,ds(Q).$$

If we let $D(\partial_x^X,\partial_y^X) = (I,\partial_x^X,\partial_y^X)$ where I is the identity operator we can write

(2.5.3) $\dot{u}(X) = \dot{u}(\dot{f};X) = \int_{\partial\Omega} \dot{f}(Q)K(\partial_x^Q,\partial_y^Q)^T D(\partial_x^X,\partial_y^X)F(X-Q)\,ds(Q).$

We note that $K^T D$ is a 3×3 matrix of differential operators. We also note that $u(X)$ is biharmonic for $X \notin \partial\Omega$ since F is a fundamental solution for the biharmonic operator Δ^2.

In order to analyze the multiple layer potential near the boundary it is necessary to represent $u(X)$ and more generally $\dot{u}(X)$ in different coordinate systems. Given $P \in \partial\Omega$, let (t_p,n_p) denote coordinates with positive axes in the directions of the unit tangent and unit inner normal vectors at P respectively. For the moment, assume $\dot{f}$ is the restriction to $\partial\Omega$ of a C^2 function $\tilde{f}$ and its x and y derivatives. Fix $X \notin \partial\Omega$ and assume $\tilde{f}$ vanishes in a neighborhood of X. Then

$$\int\int_{\Omega} \tilde{f}(Q)\Delta^2 F(X-Q)\,dxdy(Q) = 0$$

and so by Green's formula (2.22) and by (2.25) we have,

(2.5.4) $u(X) = u(\dot{f};X)$

$$= \mathrm{Re}\int\int_{\Omega} M(\partial_x,\partial_y)\tilde{f}(Q)\bar{M}(\partial_x,\partial_y)F(X-Q)\,dxdy(Q)$$

$$= \mathrm{Re}\int\int_{\Omega} M(\partial_{tp},\partial_{np})\tilde{f}(Q)\bar{M}(\partial_{tp},\partial_{np})F(X-Q)\,d_{tp}d_{np}(Q)$$

$$= \int_{\partial\Omega} (f(Q),f_{tp}(Q),f_{np}(Q))K(\partial_{tp}^Q,\partial_{np}^Q)^T F(X-Q)\,ds(Q).$$

Let $\vec{t}_P = t_1(P)\vec{i} + t_2(P)\vec{j}$. We define the 3×3 matrix $\mathcal{R}(P)$ by

$$\mathcal{R}(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_1(P) & t_2(P) \\ 0 & -t_2(P) & t_1(P) \end{pmatrix}.$$

Then $(f(Q),f_{t_p}(Q),f_{np}(Q)) = \dot{f}(Q)\mathcal{R}(P)^T$. Thus (2.5.4) becomes

(2.5.5) $u(X) = \int_{\partial\Omega} \dot{f}(Q)\mathcal{R}(P)^T K(\partial_{tp}^Q,\partial_{np}^Q)^T F(X-Q)\,ds(Q).$

We shall denote this representation of $u(X)$ in terms of the coordinates (t_P,n_P) by $u(X;\vec{n}_p)$. In addition to this representation for $u(X)$ we introduce the vector function

$$\dot{u}(X;\vec{n}_P) = \dot{u}(\dot{f};X;\vec{n}_P) = (u(X),\partial_{tp}u(X),\partial_{np}u(X)).$$

Letting $D(\partial_{tp},\partial_{np}) = (I,\partial_{tp},\partial_{np})$ we have

$$(2.5.6)\quad \dot{u}(X;\vec{n}_P) = \int_{\partial\Omega} \dot{f}(Q)\mathcal{R}(P)^T K(\partial^Q_{t_P},\partial^Q_{n_P})^T D(\partial^X_{t_P}\partial^X_{n_P})F(X-Q)\,ds(Q).$$

We recall from (4.11) of Agmon [1] that if $\dot{f}$ is the restriction to $\partial\Omega$ of a polynomial of degree ≤ 1 and its partial derivatives, the multiple layer potential $u(\dot{f};X)$ can be calculated explicitly. If $\dot{f} = (f, f_x, f_y)$ where f is a polynomial of degree ≤ 1, one has

$$(2.5.7)\qquad u(\dot{f};X) = \begin{cases} 2f(X) & \text{if } X \in \Omega \\ 0 & \text{if } X \in \mathbf{R}^2\setminus\bar{\Omega}. \end{cases}$$

This follows from a standard argument based upon Green's formula (2.2.2).

We conclude this section with an observation concerning the principal part of the multiple layer potential $u_1(X)$—the part which comes from f. From (2.2.2) we see that

$$(2.5.8)\quad u_1(X) = \int_{\partial\Omega} f(Q)\left\{\frac{dy}{ds}(Q)(-\partial^Q_{xxx} + \partial^Q_{xxy})F(X-Q)\right.$$
$$\left. + \frac{dx}{ds}(\partial^Q_{yyy} + 3\,\partial^Q_{xxy})F(X-Q)\right\}ds(Q)$$
$$= \int_{\partial\Omega} f(Q)\frac{dy}{ds}\left\{-\partial^Q_x(\partial^Q_{xx} + \partial^Q_{yy})F(X-Q)\right.$$
$$+ \frac{dx}{ds}(Q)\,\partial^Q_y(\partial^Q_{xx} + \partial^Q_{yy})F(X-Q) + 2\frac{dx}{ds}(Q)\,\partial^Q_x(\partial^Q_{xy})F(X-Q)$$
$$\left. + 2\frac{dy}{ds}(Q)\,\partial^Q_y(\partial^Q_{xy})F(X-Q)\right\}ds(Q)$$
$$= \int_{\partial\Omega} f(Q)\{\partial_{n_Q}(\Delta F)(X-Q) + 2\,\partial_{t_Q}(\partial^Q_{xy})F(X-Q)\}ds(Q)$$

where ∂_{n_Q} and ∂_{t_Q} denote the interior normal and tangential derivatives at Q. Since ΔF is a fundamental solution for Laplace's equation, the first integral in $u_1(X)$,

$$\int_{\partial\Omega} f(Q)\{\partial_{n_Q}(\Delta F)(X-Q)\,ds(Q)\},$$

is a multiple of the classical double layer potential. Thus in analyzing $u_1(X)$ we will use many of the properties of the double layer potential established in [4]. We note that the results stated in [4] are for dimensions $n \geq 3$ but the results which we shall use hold equally well for the case $n = 2$.

3. Boundary Integrals

3.1 Calderón's theorem. In Section 2.5 we introduced the multiple layer potential

$$u(X) = u(\dot{f};X) = \int_{\partial\Omega} \dot{f}(Q)K(\partial_x^Q,\partial_y^Q)^T F(X-Q)\,ds(Q)$$

where $\dot{f}$ is a compatible triple in $\mathcal{B}_p$ and $X \notin \partial\Omega$. More generally, we consider

$$\dot{u}(X) = \dot{u}(\dot{f};X) = (u(X),u_x(X),u_y(X)) \tag{3.1.1}$$

$$= \int_{\partial\Omega} \dot{f}(Q)K(\partial_x^Q,\partial_y^Q)^T D(\partial_x^X,\partial_y^X)F(X-Q)\,ds(Q).$$

We recall from (2.5.3) that $K^TD = (k_{ij})$ is a 3×3 matrix. For $X \notin \partial\Omega$ the integral in (3.1.1) exists and of course is C^∞ as a function of X. In Chapter 4 we will show that $\dot{u}(X) \to (I + \mathcal{K})\dot{f}(P)$ as X approaches P non-tangentially. The main result of this chapter is that the boundary operator $\mathcal{K}$ exists in the principal valued sense a.e. and that the mapping $\dot{f} \to \mathcal{K}\dot{f}$ is compact on the space $\mathcal{B}_p$. The main tool for proving these results is the following:

(3.1.2) Theorem (A. P. Calderón, [2]). *Let $F(z)$ be analytic in $|z| < R$, $R > 0$. Let ϕ be a real valued Lipschitz function on $\mathbf{R}$. For $\varepsilon > 0$, and $f \in L^p(\mathbf{R})$, $1 \le p < \infty$ let*

$$L_\varepsilon f(t) = \int_{|s-t|>\varepsilon} \frac{f(s)}{s-t} F\left(\frac{\phi(s)-\phi(t)}{s-t}\right) ds.$$

Then there exists an absolute constant α_0 such that $\|\phi'\|_\infty \le R\alpha_0(1+\alpha_0)^{-1/2}$ implies the operator $L^*f(t) = \sup_{\varepsilon>0} |L_\varepsilon f(t)|$ is of weak type $(1,1)$ and of strong type (p,p), $1 < p < \infty$. Moreover, $Lf(t) = \lim_{\varepsilon\to 0} L_\varepsilon f(t)$ exists almost everywhere for $f \in L^p(\mathbf{R})$, $1 \le p < \infty$.

At various points in this paper we also make use of a more general version of Calderón's theorem. We refer the reader to Theorem 4 of [3] for the precise statement of this theorem.

3.2 Trace of the multiple layer potential. In this section we analyze (principal valued) integrals over $\partial\Omega$ which arise in describing the non-tangential limits of the multiple layer potential. Let $P_0 \in \partial\Omega$ be fixed and assume $\dot{f}$ has support in $B(P_0;\delta) \cap \partial\Omega$ where $\delta > 0$. When working on $\partial\Omega$ we will use the notations $L^p(B(P_0,\delta))$ and $L_1^p(B(P_0,\delta))$ in place of $L^p(B(P_0,\delta) \cap \partial\Omega)$ and $L_1^p(B(P_0,\delta) \cap \partial\Omega)$. From (2.5.5) we have

$$u(X) = u(X;\vec{n}_{P_0}) = \int_{\partial\Omega} \dot{f}(Q)\mathcal{R}(P_0)^T K(\partial_{t_{P_0}}^Q,\partial_{n_{P_0}}^Q)^T F(X-Q)\,ds(Q). \tag{3.2.1}$$

Formally, letting $X \to P \in \partial\Omega$ in (3.2.1) we obtain integrals of the type

$$(3.2.2)\quad S(\dot{f},P;P_0) = \int_{\partial\Omega} \dot{f}(Q)\mathcal{R}(P_0)^T K(\partial^Q_{t_{P_0}},\partial^Q_{n_{P_0}})^T F(X-Q)\,ds(Q)$$

$$= \int_{\partial\Omega} f(Q)\{\langle \vec{n}_{P_0},\vec{t}_Q\rangle A_1(\partial^Q_{t_{P_0}},\partial^Q_{n_{P_0}})$$

$$+ \langle \vec{t}_{P_0},\vec{t}_Q\rangle C_1(\partial^Q_{t_{P_0}},\partial^Q_{n_{P_0}})\}F(P-Q)\,ds(Q)$$

$$+ \int_{\partial\Omega} (t_1(P_0)g(Q) + t_2(P_0)h(Q))\{\langle \vec{n}_{P_0},\vec{t}_Q\rangle A_2(\partial^Q_{t_{P_0}},\partial^Q_{n_{P_0}})$$

$$+ \langle \vec{t}_{P_0},\vec{t}_Q\rangle C_2(\partial^Q_{n_{P_0}},\partial^Q_{t_{P_0}})\}F(P-Q)\,ds(Q)$$

$$+ \int_{\partial\Omega} (t_1(P_0)h(Q) - t_2(P_0)g(Q))\{\langle \vec{n}_{P_0},\vec{t}_Q\rangle A_3(\partial^Q_{t_{P_0}},\partial^Q_{n_{P_0}})$$

$$+ \langle \vec{t}_{P_0},\vec{t}_Q\rangle C_3(\partial^Q_{t_{P_0}},\partial^Q_{n_{P_0}})\}F(P-Q)\,ds(Q)$$

$$\equiv I_1 + I_2 + I_3.$$

Replacing $t_1 g + t_2 h$ and $t_1 h - t_2 g$ by arbitrary functions in $L^p(B(P_0,\delta))$ for I_2 and I_3 we will show that all these integrals exist for almost every $P \in \partial\Omega$, and that I_1 gives rise to a compact integral operator from $L^p_1(B(P_0,\delta))$ into $L^p_1(\partial\Omega)$ while I_2 and I_3 give rise to compact integral operators from $L^p(B(P_0,\delta))$ into $L^p_1(\partial\Omega)$. To prove these results we must work in local coordinates. We assume via a partition of unity that our functions are supported in $B(P_0,\delta) \cap \partial\Omega$ and that

$$B(P_0,4\delta) \cap \Omega = B(P_0,4\delta) \cap \{(x,y) \in \mathbf{R}^2 : y > \phi(x)\}$$

where $\phi \in C^1_0(\mathbf{R})$ and $\|\phi'\|_\infty$ is sufficiently small.

We begin with the following.

Lemma (3.2.3). *Let $f \in L^p(\partial\Omega)$ with support in $B(P_0,\delta)$. For $\varepsilon > 0$ and $P \in B(P_0,4\delta)$ let*

$$I_{1,\varepsilon}f(P) = \int_{\substack{B(P_0,4\delta)\\ |P-Q|>\varepsilon}} f(Q)\{\langle \vec{n}_{P_0},\vec{t}_Q\rangle A_1(\partial_{t_{P_0}},\partial_{n_{P_0}})$$

$$+ \langle \vec{t}_{P_0},\vec{t}_Q\rangle C_1(\partial_{t_{P_0}},\partial_{n_{P_0}})\}F(P-Q)\,ds(Q).$$

Then $I_1 f(P) = \lim_{\varepsilon\to 0} I_{1,\varepsilon}f(P)$ exists for almost every $P \in B(P_0,4\delta)$ and the mapping $f \to I_1 f$ is continuous from $L^p(B(P_0,\delta))$ into $L^p(B(P_0,4\delta))$.

Proof. We first consider the operators of the form

$$J_\varepsilon f(P) = \int_{\substack{B(P_0,4\delta)\\|P-Q|>\varepsilon}} f(Q)\left(\frac{\partial^3}{\partial n_{P_0}^\alpha \partial t_{P_0}^\beta}\right)^Q F(P-Q;\vec{n}_{P_0})\,ds(Q) \qquad (\alpha+\beta=3).$$

We note that it makes no difference whether we use $F(P-Q)$ or $F(P-Q;\vec{n}_{P_0})$ in defining $J_\varepsilon f$ since $\alpha+\beta=3$. Let $J^*f(P) = \sup_{\varepsilon>0}|J_\varepsilon f(P)|$. We now show the mapping $f \to J^*f$ is bounded from $L^p(B(P_0,\delta))$ into $L^p(B(P_0,4\delta))$. In terms of the $(\vec{t}_{P_0},\vec{n}_{P_0})$ coordinate system with origin at P_0, let $P=(x,\phi(x))$ and $Q=(z,\phi(z))$. From (2.3.5) we have

$$\text{(3.2.4)}\quad \left(\frac{\partial^3}{\partial n_{P_0}^\alpha \partial t_{P_0}^\beta}\right)^Q F(P-Q,\vec{n}_{P_0})$$

$$= \operatorname{Re}\left\{\frac{1}{\pi^2}\int_\gamma \frac{(-\zeta)^\alpha d\zeta}{(\zeta^2+1)^2[(\phi(x)-\phi(z))\zeta+(x-z)]}\right\}$$

$$= \frac{1}{x-z}\operatorname{Re}\left\{\frac{1}{\pi^2}\int_\gamma \frac{(-\zeta)^\alpha d\zeta}{(\zeta^2+1)^2\left[1+\left(\frac{\phi(x)-\phi(z)}{x-z}\right)\zeta\right]}\right\}$$

$$= \frac{1}{x-z}F_\alpha\left(\frac{\phi(x)-\phi(z)}{x-z}\right)$$

where $F_\alpha(\omega) = \operatorname{Re}(1/\pi^2)\int_\gamma -\zeta^\alpha/(\zeta^2+1)^2(1+\zeta\omega)$. Thus the result for J^*f reduces to showing boundedness on $L^p(\mathbf{R})$ of the Euclidean operator

$$\text{(3.2.5)}\qquad \sup_{\varepsilon>0}\left|\int_{(x-z)^2+(\phi(x)-\phi(z))^2>\varepsilon^2} F_\alpha\left(\frac{\phi(x)-\phi(z)}{x-z}\right)\frac{f(z)}{x-z}\,dz\right|$$

where $\phi \in C_0^1(\mathbf{R})$ and $\|\phi'\|_\infty \le m_0$. We fix the contour γ to be the circle $|\zeta - i| = 1/4$. It is then easy to see that F_α is analytic in a neighborhood of 0 and hence by making m_0 small enough, we may assume

$$\left|F_\alpha\left(\frac{\phi(x)-\phi(z)}{x-z}\right)\right| \le B$$

where B is a fixed constant. Now

$$\{z:(x-z)^2+(\phi(x)-\phi(z))^2>\varepsilon^2\} = \{z:|x-z|>\varepsilon(1+m_0^2)^{-1/2}\}\setminus$$
$$\{z:|x-z|>\varepsilon(1+m_0^2)^{-1/2} \text{ and } (x-z)^2+(\phi(x)-\phi(z))^2\le\varepsilon^2\}.$$

Thus the operator in (3.2.5) is dominated by

$$\text{(3.2.6)}\quad \sup_{\varepsilon>0}\left|\int_{|x-z|>\varepsilon} F_\alpha\left(\frac{\phi(x)-\phi(z)}{x-z}\right)\frac{f(z)}{x-z}\,dz\right|$$

$$+ B \sup_{\varepsilon>0} \int_{\varepsilon(1+m_0^2)^{-1/2}\leq|x-z|\leq\varepsilon} \left|\frac{f(z)}{x-z}\right| dz$$

$$\leq \sup_{\varepsilon>0} \left| \int_{|x-z|>\varepsilon} F_\alpha\left(\frac{\phi(x)-\phi(z)}{x-z}\right) \frac{f(z)}{x-z}\, dz \right|$$

$$+ B(1+m_0^2)^{1/2} \sup_{\varepsilon>0} \varepsilon^{-1} \int_{|x-z|\leq\varepsilon} |f(z)|\,dz.$$

The last term in (3.2.6) is a constant times the Hardy-Littlewood maximal function which is bounded on $L^p(\mathbf{R})$. The first term in (3.2.6) is bounded on $L^p(\mathbf{R})$ by Calderón's theorem providing m_0 is sufficiently small depending only on the radius of convergence of the function F_α about zero. It now follows that $f \to J^*f$ is bounded from $L^p(B(P_0,\delta))$ into $L^p(B(P_0,4\delta))$. From this it is clear that the same result holds for $I_1^*f(P) \equiv \sup_{\varepsilon>0}|I_{1,\varepsilon}f(P)|$.

In order to prove $I_1f(P)$ exists for almost every $P \in B(P_0,4\delta)$ we prove that $I_1f(P)$ exists for all $P \in B(P_0,4\delta)$ when $f \in C^1(\partial\Omega)$. We first show that

$$(3.2.7)\quad \lim_{\varepsilon\to 0} \int_{\substack{\partial\Omega\\ |P-Q|>\varepsilon}} \{\langle \vec{n}_{P_0},\vec{t}_Q\rangle A_1(\partial^Q_{t_{P_0}},\partial^Q_{n_{P_0}}) + \langle \vec{t}_{P_0},\vec{t}_Q\rangle C_1(\partial^Q_{t_{P_0}},\partial^Q_{n_{P_0}})\} F(P-Q;\vec{n}_{P_0})\,ds(Q) = 1.$$

By (2.5.8) we rewrite (3.2.7) as

$$(3.2.8)\quad \lim_{\varepsilon\to 0} \int_{\substack{\partial\Omega\\ |P-Q|>\varepsilon}} \partial_{n_Q}\Delta^Q F(P-Q;\vec{n}_{P_0})\,ds(Q) + 2\lim_{\varepsilon\to 0} \int_{\substack{\partial\Omega\\ |P-Q|>\varepsilon}} \partial_{t_Q}(\partial^Q_{n_{P_0}}\partial^Q_{t_{P_0}}) F(P-Q;\vec{n}_{P_0})\,ds(Q).$$

The first limit in (3.2.8) is 1 since the kernel of the first integral is that of the double layer potential (see page 170 of [4] but note that our fundamental solution is twice the one used in their paper.) For the second integral in (3.2.8) we integrate and must compute

$$\lim_{\varepsilon\to 0}\{(\partial^Q_{n_{P_0}}\partial^Q_{t_{P_0}})F(P-Q;\vec{n}_{P_0})|_{Q=Q_1(\varepsilon)} + (\partial^Q_{n_{P_0}}\partial^Q_{t_{P_0}})F(P-Q;n_{P_0})|_{Q=Q_2(\varepsilon)}\}$$

where $Q_1(\varepsilon)$ and $Q_2(\varepsilon)$ are on $\partial\Omega$ on opposite sides of P with $|P - Q_j(t)| = \varepsilon$, $j = 1, 2$. Suppose $P = (x,\phi(x))$ and $Q_1(\varepsilon) = (x + h,\phi(x + h))$, $h > 0$, in local coordinates at P_0. From (2.3.5) we have

$$(3.2.9)\qquad \partial^Q_{n_{P_0}}\partial^Q_{t_{P_0}}F(P-Q;\vec{n}_{P_0}) = \frac{\langle P-Q,\vec{t}_{P_0}\rangle\langle P-Q,\vec{n}_{P_0}\rangle}{|P-Q|^2}.$$

In terms of local coordinates $\partial^Q_{n_{P_0}} \partial^Q_{t_{P_0}} F(P - Q; n_{P_0})|_{Q=Q_1(\varepsilon)}$ becomes

$$(3.2.10) \quad \frac{h(\phi(x) - \phi(x+h))}{h^2 + (\phi(x+h) - \phi(x))^2} = -\left(\frac{\phi(x+h) - \phi(x)}{h}\right)\left(1 + \left(\frac{\phi(x+h) - \phi(x)}{h}\right)^2\right)^{-1}.$$

Taking the limit as $h \to 0$ in (3.2.10) we find that

$$(3.2.11) \quad \lim_{\varepsilon \to 0} (\partial^Q_{n_{P_0}} \partial^Q_{t_{P_0}} F(P - Q; \vec{n}_{P_0})|_{Q_1(\varepsilon)}) = -\phi'(x)(1 + \phi'(x)^2)^{-1}.$$

Repeating this argument for $Q_2(\varepsilon) = (x + h, \phi(x + h))$ with $h < 0$, we find that

$$\lim_{\varepsilon \to 0} (\partial^Q_{n_{P_0}} \partial^Q_{t_{P_0}} F(P - Q; \vec{n}_{P_0})|_{Q_2(\varepsilon)}) = -\phi'(x)(1 + \phi'(x)^2)^{-1}.$$

It now follows that the second limit in (3.2.8) is 0. This establishes (3.2.7). Now suppose $f \in C^1(\partial\Omega)$ with support in $B(P_0, 4\delta)$. We write

$$(3.2.12) \quad I_{1,\varepsilon} f(P) = \int_{\substack{\partial\Omega \\ |P-Q|>\varepsilon}} (f(Q) - f(P))\{\langle \vec{n}_{P_0}, \vec{t}_Q \rangle A_1(\partial_{t_{P_0}}, \partial_{n_{P_0}}) + \langle \vec{t}_{P_0}, \vec{t}_Q \rangle C_1(\partial_{t_{P_0}}, \partial_{n_{P_0}})\} F(P - Q; \vec{n}_{P_0})\, ds(Q)$$
$$+ f(P) \int_{\substack{\partial\Omega \\ |P-Q|>\varepsilon}} \{\langle \vec{n}_{P_0}, \vec{t}_Q \rangle A_1(\partial_{t_{P_0}}, \partial_{n_{P_0}}) + \langle \vec{t}_{P_0}, \vec{t}_Q \rangle C_1(\partial_{t_{P_0}}, \partial_{n_{P_0}})\} F(P - Q; \vec{n}_{P_0})\, ds(Q).$$

By estimates in §9 of [1] we have $|A_1(\partial^Q_{t_{P_0}}, \partial^Q_{n_{P_0}}) F(P - Q; n_{P_0})| = O(|P - Q|^{-1})$ and $|C_1(\partial^Q_{t_{P_0}}, \partial^Q_{n_{P_0}}) F(P - Q); n_{P_0})| = O(|P - Q|^{-1})$. Thus for $f \in C^1(\partial\Omega)$ the first integral in (3.2.12) converges absolutely as $\varepsilon \to 0$ while by (3.2.7) the second integral in (3.2.12) converges to $f(P)$ as $\varepsilon \to 0$. Thus $I_1 f(P)$ exists for all P if $f \in C^1(\partial\Omega)$. By standard arguments, it now follows from the maximal estimate that for $f \in L^p(B(P_0, \delta))$, $I_1 f(P)$ exists for almost every P in $B(P_0, 4\delta)$. The continuity of $f \to I_1 f$ also readily follows from the estimate for the maximal function. This completes the proof of Lemma 3.2.3.

We now establish compactness of $f \to I_1 f$ from the space $L^p_1(\partial\Omega)$ into itself. We have

Lemma 3.2.13. *Let $f \in L^p_1(\partial\Omega)$, $1 < p < \infty$ with support in $B(P_0, \delta) \cap \partial\Omega$. Let $\psi \in C^1_0(\partial\Omega)$ with support in $B(P_0, 4\delta) \cap \partial\Omega$. Then the mapping $f \to \psi I_1 f$ is compact from $L^p_1(B(P_0, \delta))$ into $L^p_1(B(P_0, 4\delta))$.*

Proof. By (2.5.8) we again write $I_1 f(P)$ as

$$\text{(3.2.14)} \quad \text{p.v.} \int_{\partial\Omega} f(Q)\,\partial_{n_Q}\Delta^Q F(P-Q;\vec{n}_{P_0})\,ds(Q) + 2\,\text{p.v.} \int_{\partial\Omega} f(Q)\,\partial_{t_Q}(\partial^Q_{nP_0}\partial^Q_{tP_0})F(P-Q;\vec{n}_{P_0})\,ds(Q).$$

Since the first integral in (3.2.14) is the double layer potential of f, the result of the lemma holds for this integral by Theorem 1.6 of [4]. Thus we need only prove the lemma for the second integral in (3.2.14). Letting $P = (x,\phi(x))$ and $Q = (z,\phi(z))$ in local coordinates with origin at P_0 and noting (3.2.9) this integral becomes

$$\text{(3.2.15)} \quad \text{p.v.} \int_{\mathbf{R}} f(z)\frac{\partial}{\partial z}\left\{\frac{(x-z)(\phi(x)-\phi(z))}{(x-z)^2+(\phi(x)-\phi(z))^2}\right\}dz = \text{p.v.}\int_{\mathbf{R}} f(z)\,\Psi(x,z)\,dz$$

where $\phi \in C^1_0(\mathbf{R})$. We note that in changing from principal valued integrals on $\mathbf{R}$, an error term arises. This error term is estimated just as in Theorem 1.2 of [4]. We first note that $\int_{\mathbf{R}}\Psi(x,z)\,dz = 0$. A simple computation shows that

$$\int_x^\infty \Psi(x,z)\,dz = \frac{-\phi'(x)}{1+(\phi'(x))^2} = -\int_{-\infty}^x \Psi(x,z)\,dz.$$

We denote the integral in (3.2.15) by $Sf(x)$. We now show that for $f \in L^p_1(\mathbf{R})$ when $\phi \in C^\infty_0(\mathbf{R})$, $(d/dx)Sf(x)$ exists and is given by

$$\text{(3.2.16)} \qquad \frac{d}{dx}Sf(x) = \int_{\mathbf{R}} (f(z)-f(x))\,\partial_x\Psi(x,z)\,dz.$$

We first consider $f \in C^\infty_0(\mathbf{R})$. For $h > 0$ we have

$$\text{(3.2.17)} \quad \begin{aligned} \frac{1}{h}[Sf(x+h)-Sf(x)] &= \frac{1}{h}\int_{\mathbf{R}} [f(z)-f(x)][\psi(x+h,z)-\psi(x,z)]\,dz \\ &= \left(\int_{|x-z|\ge 2h} + \int_{|x-z|\le 2h}\right)(\ldots)\,dz \\ &\equiv A_h(x) + B_h(x). \end{aligned}$$

Now $B_h(x) \to 0$ as $h \to 0$. To see this, we note that

$$|\Psi(x+h,z)-\Psi(x,z)| \le |\Psi(x+h,z)| + |\Psi(x,z)| \le C$$

where C depends on ϕ but not on h. By the mean value theorem

$$\text{(3.2.18)} \quad \begin{aligned} |B_h(x)| &\le \frac{1}{h}\int_{|x-z|\le 2h} \|f'\|_\infty |\Psi(x+h,z)-\Psi(x,z)|\,dz \\ &\le 2\,ch \to 0 \qquad \text{as } h \to 0. \end{aligned}$$

To show $A_h(x) \to \int_{\mathbf{R}} (f(z) - f(x))\Psi_x(x,z)\,dz$ we apply the dominated convergence theorem. A laborious computation shows that

$$\left|\frac{1}{h}(\Psi(x+h,z) - \Psi(x,z))\right| \le C|x-z|^{-2}$$

for $|x - z| \ge 1$ while for $2h \le |x - z| \le 1$,

$$\left|\frac{1}{h}(\Psi(x+h,z) - \Psi(x,z))\right| \le C|x-z|^{-1}.$$

Applying the mean value theorem to f for $2h \le |x - z| \le 1$ shows that on this interval the integrand in A_h is uniformly bounded. Thus by the dominated convergence theorem, we obtain (3.2.16) when f and ϕ are in $C_0^\infty(\mathbf{R})$.

In order to extend (3.2.16) to the case when $f \in L_1^p(\mathbf{R})$, we note that a simple calculation shows that for $\phi \in C_0^\infty$, $|\Psi_x(x,z)| \le C|x-z|^{-1}$ where C depends only upon ϕ. This implies

$$\left\|\frac{d}{dx} Sf\right\|_p^p < c \int\int_D \left|\frac{f(x) - f(z)}{x - z}\right|^p dx\,dz \le C'\|f'\|_p^p \tag{3.2.19}$$

where $D = \operatorname{supp}\phi \times \operatorname{supp}\phi$ and C' still depends upon ϕ. This shows that when $\phi \in C_0^\infty(\mathbf{R})$, Sf maps $L_1^p(\mathbf{R})$ into itself continuously and that the formula for $(d/dx)Sf$ remains valid for $f \in L_1^p(\mathbf{R})$.

Finally, we extend (3.2.16) to the case when $\phi \in C_0^1(\mathbf{R})$. In this case $(d/dx)Sf$ is interpreted in the sense of distributions. Choose $\{\phi_j\}$ with $\phi_j \in C_0^\infty(\mathbf{R})$ and $\phi_j \to \phi$, $\phi_j' \to \phi'$ uniformly as $j \to \infty$ with $\|\phi_j'\|_\infty$ sufficiently small for all j. Set

$$S_j f(x) \equiv \int_{\mathbf{R}} \Psi_j(x,z)(f(x) - f(z))\,dz, \tag{3.2.20}$$

where ψ_j is as in (3.2.15) with ϕ_j replacing ϕ. Now

$$\frac{d}{dx} S_j f(x) = \int_{\mathbf{R}} \frac{\partial}{\partial x} \Psi_j(x,z)(f(x) - f(z))\,dz.$$

Writing out $(\partial/\partial x)\psi_j(x,z)$ it is easy to check that Calderón's theorem [3], Theorem 4 can be applied to conclude

$$\left\|\frac{d}{dx} S_j f\right\|_{L^p(\mathbf{R})} \le C\|f\|_{L_1^p(\mathbf{R})}$$

where C does not depend upon j. Since $S_j f \to Sf$ in $L^p(\mathbf{R})$ as $j \to \infty$, we conclude that $Sf \in L_1^p(\mathbf{R})$ whenever $f \in L_1^p(\mathbf{R})$ and that $\|Sf\|_{L_1^p(\mathbf{R})} \le C\|f\|_{L_1^p(\mathbf{R})}$. Finally in the sense of distributions, we have

$$\frac{d}{dx} Sf = \lim_j \frac{d}{dx} S_j f = \lim_j \int_{\mathbf{R}} \frac{\partial}{\partial x} \Psi_j(x,z)(f(x) - f(z))\,dz$$

$$= \text{p.v.} \int \frac{\partial}{\partial x} \Psi(x,z)(f(x) - f(z))\,dz.$$

Thus (3.2.16) is now established when $\phi \in C_0^1(\mathbf{R})$.

We now turn to the final step of the lemma which is to prove that the mapping $f \to \psi Sf$ is compact on $L_1^p(I)$ where I is a fixed interval containing supports of ϕ and ψ. Let $P_2(\phi) = P_2(\phi;x,z) = \phi(x) - \phi(z) - \phi'(z)(x - z)$, the second order Taylor series remainder of ϕ at x expanded about z. Again let $\{\phi_j\}$ be a sequence of functions in $C_0^\infty(\mathbf{R})$ all with support in I such that $\phi_j \to \phi$ and $\phi_j' \to \phi'$ uniformly. Then using the formula for $(d/dx)Sf$ we have

(3.2.21) $\displaystyle \frac{d}{dx} Sf(x)$

$$= \text{p.v.} \int_{\mathbf{R}} \frac{[(x - z)^2 + (\phi(x) - \phi(z))^2][(x - z)^2 - (\phi(x) - \phi(z))^2]}{((x - z)^2 + (\phi(x) - \phi(z))^2)^3} \cdot [(\phi - \phi_j)'(x) - (\phi - \phi_j)'(z)](f(x) - f(z))\,dz$$

$$- 2\,\text{p.v.} \int_{\mathbf{R}} \frac{[(x - z)^3 - (\phi(x) - \phi(z))^3\phi'(x)]}{((x - z)^2 + (\phi(x) - \phi(z))^2)^3} \cdot P_2(\phi - \phi_j;x,z)(f(x) - f(z))\,dz$$

$$+ 6\,\text{p.v.} \int_{\mathbf{R}} \frac{(x - z)(\phi(x) - \phi(z))(\phi(x) - \phi(z) - \phi'(x)(x - z))}{((x - z)^2 + (\phi(x) - \phi(z))^2)^3} \cdot P_2(\phi - \phi_j;x,z)(f(x) - f(z))\,dz$$

$$+ \text{p.v.} \int_{\mathbf{R}} \frac{[(x - z)^2 + (\phi(x) - \phi(z))^2][(x - z)^2 - (\phi(x) - \phi(z))^2]}{((x - z)^2 + (\phi(x) - \phi(z))^2)^3} \cdot (\phi_j(x) - \phi_j(z))(f(x) - f(z))\,dz$$

$$- 2 \int_{\mathbf{R}} \frac{[(x - z)^3 - (\phi(x) - \phi(z))^3\phi'(x)]}{((x - z)^2 + (\phi(x) - \phi(z))^2)^3} \cdot P_2(\phi_j;x,z)(f(x) - f(z))\,dz$$

$$+ 6 \int_{\mathbf{R}} \frac{(x - z)(\phi(x) - \phi(z))(\phi(x) - \phi(z) - \phi'(x)(x - z))}{((x - z)^2 + (\phi(x) - \phi(z))^2)^3} \cdot P_2(\phi_j;x,z)(f(x) - f(z))\,dz.$$

By Calderón's theorem [3] the first three integrals in (3.2.21) as operators from $L_1^p(I)$ into $L^p(I)$ have norms tending to zero as $j \to \infty$. Since $\phi_j \in C_0^\infty$, the last three integrands are weakly singular and it is easy to see that the last three operators are compact from $L_1^p(I)$ into $L^p(I)$ for each j. It now follows that the mapping $f \to Sf$ is compact from $L_1^p(I)$ into itself. This completes the proof of Lemma (3.2.13).

Returning to (3.2.2) it now follows that the operator I_1 is compact from $L_1^p(B(P_0,\delta))$ into $L_1^p(\partial\Omega)$. Replacing $t_1g + t_2h$ and $t_1h - t_2g$ by arbitrary L^p functions with supports in $B(P_0,\delta)$ in I_2 and I_3 we now show that the operators I_2 and I_3 are compact from $L^p(B(P_0,\delta))$ into $L_1^p(\partial\Omega)$. We have

Lemma (3.2.22). *Let $f \in L^p(B(P_0,\delta))$, $1 < p < \infty$, and for $j = 2, 3$ and $P \in \partial\Omega$, let $\psi \in C_0^1(\partial\Omega)$ with support in $B(P_0,4\delta)$, and for $j = 2, 3$ let $I_jf(P)$ be defined as in (3.2.2)—i.e.*

$$(3.2.23)\quad I_jf(P) = \int_{\partial\Omega} f(Q)\{\langle \vec{n}_{P_0},\vec{t}_Q\rangle A_j(\partial^Q_{t_{P_0}},\partial^Q_{n_{P_0}}) + \langle \vec{t}_{P_0},\vec{t}_Q\rangle C_j(\partial^Q_{t_{P_0}},\partial^Q_{n_{P_0}})\}\cdot F(P-Q)\,ds(Q).$$

Then for $j = 2, 3$ the mapping $f \to \psi I_jf$ is compact from $L^p(B(P_0,\delta))$ into $L_1^p(\partial\Omega)$.

Proof. We first note that the integrals in (3.2.23) converge absolutely for all $P \in \partial\Omega$ since the A_j's and C_j's are second order differential operators and we have from (3.1) of [1]

$$\left|\left(\frac{\partial^2}{\partial n^\alpha_{P_0}\partial t^\beta_{P_0}}\right)F(P-Q)\right| = O\log|P-Q|.$$

It also readily follows from Minkowski's integral inequality that each of these operators is bounded from $L^p(B(P_0,\delta))$ into $L^p(\partial\Omega)$. We now consider operators of the form

$$(3.2.24)\quad \begin{aligned} S_\alpha f(P) &= \int_{\partial\Omega} f(Q)\left(\frac{\partial^2}{\partial n^\alpha_{P_0}\partial t^\beta_{P_0}}\right)^Q F(P-Q;\vec{n}_{P_0})\,ds(Q) \qquad (\alpha+\beta=2)\\ &= \operatorname{Re}\left\{\frac{1}{\pi^2}\int_{\partial\Omega} f(Q)\int_\gamma \frac{(-\zeta)^\alpha \log\langle P-Q,\vec{n}_{P_0}\bar{\zeta}+t_{P_0}\rangle}{(\zeta^2+1)^2}\,d\zeta\,ds(Q)\right\}.\end{aligned}$$

(We are assuming that f is real valued.) Letting $P = (x,\phi(x))$ and $Q = (z,\phi(z))$ in local coordinates with origin at P_0, our problem is to analyze

$$(3.2.25)\quad S_\alpha f(x) = \operatorname{Re}\left\{\frac{1}{\pi^2}\int_{\mathbf{R}} f(z)\int_\gamma \frac{(-\zeta)^\alpha\log[(\phi(x)-\phi(z))\zeta+(x-z)]}{(\zeta^2+1)^2}\,d\zeta\,dz\right\}$$

where $\phi \in C_0^1(\mathbf{R})$. We show that for $f \in L^p(\mathbf{R})$, $S_\alpha f$ is in $L_1^p(\mathbf{R})$. Let $\Psi(x,z)$ denote the kernel in (3.2.25). We will show that for $f \in L^p(\mathbf{R})$, $S_\alpha f$ has a distributional derivative in $L^p(\mathbf{R})$ given by

$$(3.2.26)\qquad \frac{d}{dx}S_\alpha f(x) = \operatorname{Re}\frac{1}{\pi^2}\,\text{p.v.}\int_R f(z)\,\partial_x\Psi(x,z)\,dz.$$

An integration by parts argument shows that for any test function $\beta(x)$,

$$\int_{\mathbf{R}} \beta'(x) S_\alpha f\,dx = -\int_{\mathbf{R}} \beta(x)\left\{\text{p.v.} \int_{\mathbf{R}} f(z)\,\partial_x \Psi(x,z)\,dz\right\} dx \tag{3.2.27}$$

provided the maximal operator

$$\sup_{\varepsilon>0} \left| \int_{|x-z|>\varepsilon} f(z)\,\partial_x \Psi(x,z)\,dz \right| \tag{3.2.28}$$

is bounded on $L^p(\mathbf{R})$. From (3.2.25) we have

$$\partial_x \Psi(x,z) = \phi'(x) \int_\gamma \frac{(-1)^\alpha \zeta^{\alpha+1}\,d\zeta}{(\zeta^2+1)((\phi(x)-\phi(z))\zeta + (x-z))} + \int_\gamma \frac{(-\zeta)^\alpha\,d\zeta}{(\zeta^2+1)((\phi(x)-\phi(z))\zeta + (x-z))}. \tag{3.2.29}$$

Applying Calderon's theorem to each term separately, it follows that the maximal operator in (3.2.28) is bounded on $L^p(\mathbf{R})$. It also follows that the integral in (3.2.26) exists for a.e. x and represents a function in $L^p(\mathbf{R})$. This shows that $S_\alpha f$ has a distributional derivative given by an L^p function and thus $S_\alpha f$ is in L_1^p. It now follows that the operators I_2 and I_3 are continuous from $L^p(B(P_0;\delta))$ into $L_1^p(\partial\Omega)$. The last step is to show these operators are compact. In order to prove compactness, we must work with the kernels of these operators explicitly. From (2.2.3) it suffices to prove each of the following operators is compact from $L^p(B(P_0,\delta))$ into $L^p(B(P_0,4\delta))$:

$$\psi(P) S_1 f(P) = \psi(P) \int_{\partial\Omega} f(Q) \left(\frac{\partial^2}{\partial t_{P_0}^2} - \frac{\partial^2}{\partial n_{P_0}^2}\right)^Q F(P-Q;\vec{n}_{P_0})\,ds(Q) \tag{3.2.30}$$

and

$$\psi(P) S_2 f(P) = \psi(P) \int_{\partial\Omega} f(Q) \left(\frac{\partial^2}{\partial n_{P_0} \partial t_{P_0}}\right)^Q F(P-Q;\vec{n}_{P_0})\,ds(Q).$$

We begin with $S_1 f$. From (2.3.5) we have

$$S_1 f(P) = \frac{1}{2\pi} \int_{\partial\Omega} f(Q) \frac{\langle P-Q, n_{P_0}\rangle^2}{|P-Q|^2}\,ds(Q). \tag{3.2.31}$$

Using local coordinates with origin at P_0 it suffices to prove the compactness from $L^p(I)$ into $L_1^p(I)$ of the Euclidean operator

$$\tilde{\psi}(x)\tilde{S}_1 f(x) = \tilde{\psi}(x) \int_{\mathbf{R}} f(z) \frac{(\phi(x)-\phi(z))^2}{(x-z)^2 + (\phi(x)-\phi(z))^2}\,dz \tag{3.2.32}$$

where I is a fixed interval of $\mathbf{R}$. From our previous remarks $\tilde{\psi}\tilde{S}_1 f$ has a distributional derivative given by

$$(3.2.33)\qquad \tilde{\psi}'(x)\tilde{S}_1 f(x) + \tilde{\psi}(x)\,\text{p.v.}\int_{\mathbf{R}} f(z)\frac{\partial}{\partial x}\left\{\frac{(\phi(x)-\phi(z))^2}{(x-z)^2+(\phi(x)-\phi(z))^2}\right\}dz.$$

The first term in (3.2.33) is clearly compact on $L^p(I)$ so we need only prove compactness on $L^p(I)$ for the second integral in (3.2.33). This integral is

$$(3.2.34)\qquad -2\tilde{\psi}(x)\,\text{p.v.}\int_{\mathbf{R}} f(z)$$
$$\cdot\frac{(\phi(x)-\phi(z))(x-z)[\phi(x)-\phi(z)-\phi'(x)(x-z)]}{((x-z)^2+(\phi(x)-\phi(z))^2)^2}\,dz.$$

Let $\{\phi_j\}$ be a sequence in $C_0^\infty(\mathbf{R})$ with supports in I and $\phi_j \to \phi$, $\phi_j' \to \phi'$, uniformly as $j \to \infty$. For j fixed it is easy to see that the operator

$$(3.2.35)\qquad -2\tilde{\psi}(x)\,\text{p.v.}\int_{\mathbf{R}} f(z)$$
$$\cdot\frac{(\phi(x)-\phi(z))(x-z)[\phi_j(x)-\phi_j(z)-\phi_j'(x)(x-z)]}{((x-z)^2+(\phi(x)-\phi(z))^2)^2}\,dz$$

is compact from $L^p(I)$ into itself. Denoting the operator in (3.2.34) by $\tilde{T}_1$ and the operator in (3.2.35) by $\tilde{T}_{1,j}$, we have

$$(3.2.36)\qquad (\tilde{T}_1 - \tilde{T}_{1,j})f(x)$$
$$= 2\tilde{\psi}(x)\int_{\mathbf{R}} f(x)\frac{(\phi(x)-\phi(z))(x-z)[(\phi-\phi_j)(x)-(\phi-\phi_j)(z)]}{((x-z)^2+(\phi(x)-\phi(z))^2)^2}\,dz$$
$$- 2\tilde{\psi}(x)(\phi'(x)-\phi_j'(x))\int_{\mathbf{R}} f(z)\frac{(\phi(x)-\phi(z))(x-z)^2}{((x-z)^2+(\phi(x)-\phi(z))^2)^2}\,dz.$$

By Calderón's theorem we have

$$\|(\tilde{T}_1 - \tilde{T}_{1,j})f\|_p \le C\|f\|_p\|\phi\|_\infty(\|\phi-\phi_j\|_\infty + \|\phi'-\phi_j'\|_\infty).$$

This shows that the operator in (3.2.34) is the norm limit of compact operators and hence is compact on $L^p(I)$. This establishes compactness for the first operator in (3.2.30) and we now prove compactness for the second operator in (3.2.30). Again using (2.3.5) we have

$$(3.2.37)\qquad S_2 f(P) = -\frac{1}{4\pi}\int_{\partial\Omega} f(Q)\left\{\frac{\langle P-Q,\vec{n}_{P_0}\rangle\langle P-Q,\vec{t}_{P_0}\rangle}{|P-Q|^2}\right\}ds(Q).$$

In terms of local coordinates we find $\tilde{\psi}\tilde{S}_2 f$ has a distributional derivative given by

$$(3.2.38)\quad \tilde{\psi}'(x)\tilde{S}_2f(x) + \tilde{\psi}(x)\,\text{p.v.}\int_{\mathbf{R}} f(z)\frac{[(\phi(x)-\phi(z))^2-(x-z)^2][\phi(x)-\phi(z)-\phi'(x)(x-z)]}{((x-z)^2+(\phi(x)-\phi(z))^2)^2}\,dz$$
$$\equiv \tilde{\psi}'(x)\tilde{S}_2f(x)+\tilde{T}_2f(x).$$

An argument just like the one used for T_1 shows that the mapping $f \to \tilde{T}_2 f$ is compact on $L^p(I)$. This completes the proof of Lemma (3.2.22).

We return briefly to (3.2.2). Let $\eta(P)$ be a C^1 function on $\partial\Omega$ with $\eta(P) \equiv 1$ for $|P_0 - P| \leq \delta$, $\eta(P) \equiv 0$ for $|P_0 - P| > 4\delta$, and $0 \leq \eta(Q) \leq 1$. For any of the integral operators I_jf in (3.2.2) where f has support in $B(P_0,\delta)$, we write

$$(3.2.39)\qquad I_jf(P) = \eta(P)I_jf(P) + (1-\eta(P))I_jf(P).$$

By the preceding lemmas, the first operator in (3.2.39) is compact from $L_1^p(B(P_0,\delta))$ or $L^p(B(P_0,\delta))$ into $L_1^p(\partial\Omega)$. The second operator in (3.2.39) is compact from $L_1^p(B(P_0,\delta))$ or $L^p(B(P_0,\delta))$ into $L_1^p(\partial\Omega)$ because support of f and support of $(1-\eta)$ are separated by a positive distance and the kernels are smooth off the diagonal. It now follows that the operator $\dot{f} \to S(f,P;P_0)$ is compact from $L_1^p(B(P_0,\delta)) \times L^p(B(P_0,\delta)) \times L^p(B(P_0,\delta))$ into $L_1^p(\partial\Omega)$.

Putting the preceding results together we have the following

Theorem 3.2.40. *Let $\dot{f} \in L_1^p(\partial\Omega) \times L^p(\partial\Omega) \times L^p(\partial\Omega)$ and let $\{B(P_j,\delta_j)\}_{j=1}^n$ be a finite covering of $\partial\Omega$ such that*

$$B(P_j,4\delta_j) \cap \Omega = B(P_j,4\delta_j) \cap \{(x,y) : y > \phi_j(x), \phi_j \in C_0^1(\mathbf{R})\}.$$

Let $\{\psi_j\}$ be a smooth partition of unity subordinate to $\{B(P_j,\delta_j)\}_{j=1}^n$. Define

$$(3.2.41)\qquad T\dot{f}(P) = \sum_{j=1}^{n} S(\dot{f}_j,P;P_j)$$

where $\dot{f}_j = (\psi_j f, f(\partial\psi_j/\partial x) + \psi_j g, f(\partial\psi_j/\partial y) + \psi_j h)$ and $S(\cdot,P;P_j)$ are defined in (3.2.2). Then for $\dot{f} \in \mathcal{B}_p(\partial\Omega)$ and δ_j sufficiently small one has $T\dot{f} \in L_1^p(\partial\Omega)$ and

$$(3.2.42)\qquad \|T\dot{f}\|_{L_1^p(\partial\Omega)} \leq c\|\dot{f}\|_{p,1}.$$

Moreover, the mapping $\dot{f} \to T\dot{f}$ is compact from $L_1^p(\partial\Omega) \times L^p(\partial\Omega) \times L^p(\partial\Omega)$ into $L_1^p(\partial\Omega)$.

We conclude this section by noting that the operator given by (3.2.41) is closely related to the boundary values of the multiple layer potential $u(\dot{f};X)$.

3.3. Trace of the derivatives of the multiple layer potential. In this section we analyze (principal valued) integrals over $\partial\Omega$ which arise in the non-tangential limits of the derivatives of the multiple layer potential.

As in the previous section we must first work locally. As before, we let $P_0 \in \partial\Omega$ and assume $\dot{f}$ has support in $B(P_0,\delta) \cap \partial\Omega$. As observed by Agmon [1], for $X \in B(P_0,4\delta) \cap \Omega$ it is convenient to use a representation for $u(X)$ which varies with $P \in B(P_0,4\delta) \cap \partial\Omega$. For $X \in B(P_0,4\delta) \cap \partial\Omega$ and $P \in B(P_0,4\delta) \cap \partial\Omega$ we have

$$(3.3.1) \quad u(X) = u(X;\vec{n}_p) = \int_{\partial\Omega} \dot{f}(Q)\mathcal{R}(P)^T K(\partial_{t_p}^Q,\partial_{n_p}^Q)^T F(X-Q)\,ds(Q).$$

Differentiating u in directions parallel to $\vec{t}_P$ and $\vec{n}_P$ we have

$$(3.2.2) \quad \begin{aligned} \partial_{t_p}u(f;X) &= \int_{\partial\Omega} f(Q)\mathcal{R}(P)^T K(\partial_{t_p}^Q,\partial_{n_p}^Q)^T \partial_{t_p}^X F(X-Q)\,ds(Q) \\ \partial_{n_p}u(f;X) &= \int_{\partial\Omega} f(Q)\mathcal{R}(P)^T K(\partial_{t_p}^Q,\partial_{n_p}^Q)^T \partial_{n_p}^X F(X-Q)\,ds(Q). \end{aligned}$$

Let $\tilde{f}(P)$ denote the constant vector $(f(P),0,0)$. We know from (2.5.7) that $u(\tilde{f}(P);X) = 2f(P)$ and consequently $\partial_{t_p}u(\tilde{f}(P);X) = \partial_{n_p}u(\tilde{f}(P):X) = 0$. Writing out $\partial_{t_p}u(\dot{f}-\tilde{f}(P);X)$ and $\partial_{n_p}u(f-\tilde{f}(P);X)$ and then formally letting $X \to P$ we obtain

$$(3.3.3) \quad \partial_{t_p}u(\dot{f};P)$$

$$\begin{aligned} &= \int_{\partial\Omega} (\dot{f}(Q)-\tilde{f}(P))\mathcal{R}(P)^T K(\partial_{t_p}^Q,\partial_{n_p}^Q)^T \partial_{t_p}^P F(P-Q)\,ds(Q) \\ &= \int_{\partial\Omega} (f(Q)-f(P))\{\langle\vec{n}_P,\vec{t}_Q\rangle A_1(\partial_{t_p}^Q,\partial_{n_p}^Q) + \langle\vec{t}_P,\vec{t}_Q\rangle C_1(\partial_{t_p}^Q,\partial_{n_p}^Q)\}\partial_{t_p}^P F(P-Q)\,ds(Q) \\ &\quad + \int_{\partial\Omega} (t_1(P)g(Q)+t_2(P)h(Q))\{\langle\vec{n}_P,\vec{t}_Q\rangle A_2(\partial_{t_p}^Q,\partial_{n_p}^Q) \\ &\qquad\qquad + \langle\vec{t}_P,\vec{t}_Q\rangle C_2(\partial_{t_p}^Q,\partial_{n_p}^Q)\}\partial_{t_p}^P F(P-Q)\,ds(Q) \\ &\quad + \int_{\partial\Omega} (t_1(P)h(Q)-t_2(P)g(Q))\{\langle\vec{n}_P,\vec{t}_Q\rangle A_3(\partial_{t_p}^Q,\partial_{n_p}^Q) \\ &\qquad\qquad + \langle\vec{t}_P,\vec{t}_Q\rangle C_3(\partial_{t_p}^Q,\partial_{n_p}^Q)\}\partial_{t_p}^P F(P-Q)\,ds(Q) \end{aligned}$$

and

$$\begin{aligned} (3.3.4) \quad \partial_{n_p}u(\dot{f};P) &= \int_{\partial\Omega} (\dot{f}(Q)-\tilde{f}(P))\mathcal{R}(P)^T K(\partial_{t_p}^Q,\partial_{n_p}^Q)^T \partial_{n_p}^P F(P-Q)\,ds(Q) \\ &= \int_{\partial\Omega} (f(Q)-f(P))\{\langle\vec{n}_P,\vec{t}_Q\rangle A_1(\partial_{t_p}^Q,\partial_{n_p}^Q) \\ &\qquad\qquad + \langle\vec{t}_P,\vec{t}_Q\rangle C_1(\partial_{t_p}^Q,\partial_{n_p}^Q)\}\partial_{n_p}^P F(P-Q)\,ds(Q) \\ &\quad + \int_{\partial\Omega} (t_1(P)g(Q)+t_2(P)h(Q))\{\langle\vec{n}_P,\vec{t}_Q\rangle A_2(\partial_{t_p}^Q,\partial_{n_p}^Q) \end{aligned}$$

$$+ \langle \vec{t}_P, \vec{t}_Q \rangle C_2(\partial^Q_{t_P}, \partial^Q_{n_P})\} \partial^P_{n_P} F(P - Q)\, ds(Q)$$

$$+ \int_{\partial\Omega} (t_1(P)h(Q) - t_2(P)g(Q))\{\langle \vec{n}_P, \vec{t}_Q \rangle A_3(\partial^Q_{t_P}, \partial^Q_{n_P})$$

$$+ \langle \vec{t}_P, \vec{t}_Q \rangle C_3(\partial^Q_{t_P}, \partial^Q_{n_P})\} \partial^P_{n_P} F(P - Q)\, ds(Q).$$

We must analyze the integrals in (3.3.3) and (3.3.4) individually. We first analyze integrals of the following types as integral operators from $L^p_1(B(P_0,\delta))$ into $L^p(\partial\Omega)$:

$$(3.3.5)\quad \begin{aligned} J_1 f(P) &= \int_{\partial\Omega} (f(Q) - f(P)) A_1(\partial^Q_{t_P}, \partial^Q_{n_P}) \partial^P_{t_P} F(P - Q) \langle \vec{n}_P, \vec{t}_Q \rangle\, ds(Q) \\ J_2 f(P) &= \int_{\partial\Omega} (f(Q) - f(P)) A_1(\partial^Q_{t_P}, \partial^Q_{n_P}) \partial^P_{n_P} F(P - Q) \langle \vec{n}_P, \vec{n}_Q \rangle\, ds(Q) \\ J_3 f(P) &= \int_{\partial\Omega} (f(Q) - f(P)) C_1(\partial^Q_{t_P}, \partial^Q_{n_P}) \partial^P_{t_P} F(P - Q) \langle \vec{t}_P, \vec{t}_Q \rangle\, ds(Q) \\ J_4 f(P) &= \int_{\partial\Omega} (f(Q) - f(P)) C_1(\partial^Q_{t_P}, \partial^Q_{n_P}) \partial^P_{n_P} F(P - Q) \langle \vec{t}_P, \vec{t}_Q \rangle\, ds(Q). \end{aligned}$$

We will analyze the integrals of the following types as integral operators from $L^p(B(P_0,\delta))$ into $L^p(\partial\Omega)$.

$$(3.3.6)\quad \begin{aligned} J_5 f(P) &= \int_{\partial\Omega} f(Q) A_2(\partial^Q_{t_P}, \partial^Q_{n_P}) \partial^P_{n_P} F(P - Q) \langle \vec{n}_P, \vec{t}_Q \rangle\, ds(Q) \\ J_6 f(P) &= \int_{\partial\Omega} f(Q) A_2(\partial^Q_{t_P}, \partial^Q_{n_P}) \partial^P_{n_P} F(P - Q) \langle \vec{n}_P, \vec{t}_Q \rangle\, ds(Q) \\ J_7 f(P) &= \int_{\partial\Omega} f(Q) A_3(\partial^Q_{t_P}, \partial^Q_{n_P}) \partial^P_{t_P} F(P - Q) \langle \vec{n}_P, \vec{t}_Q \rangle\, ds(Q) \\ J_8 f(P) &= \int_{\partial\Omega} f(Q) A_3(\partial^Q_{t_P}, \partial^Q_{n_P}) \partial^P_{n_P} F(P - Q) \langle \vec{n}_P, \vec{t}_Q \rangle\, ds(Q) \\ J_9 f(P) &= \int_{\partial\Omega} f(Q) C_3(\partial^Q_{t_P}, \partial^Q_{n_P}) \partial^P_{t_P} F(P - Q) \langle \vec{t}_P, \vec{t}_Q \rangle\, ds(Q) \\ J_{10} f(P) &= \int_{\partial\Omega} f(Q) C_3(\partial^Q_{t_P}, \partial^Q_{n_P}) \partial^P_{n_P} F(P - Q) \langle \vec{t}_P, \vec{t}_Q \rangle\, ds(Q). \end{aligned}$$

We recall from (2.2.3) that $C_2 \equiv 0$. We also note that in all of the above integrals we may replace $F(P - Q)$ by $F(P - Q; \vec{n}_P)$ since all differential operators acting on F are of degree ≥ 3. We also replace

$$\partial_{t_P}^P F(P-Q;\vec{n}_P) \text{ by } (-1)\partial_{t_P}^Q F(P-Q;\vec{n}_P).$$

We begin by analyzing the operators in (3.3.5). We have

Lemma (3.3.7). *Let $f \in L_1^p(\partial\Omega)$ with support in $B(P_0,\delta)$. For $\varepsilon > 0$ and $P \in B(P_0,4\delta)$ let $J_{i,\varepsilon}f(P)$ denote any of the integrals in (3.3.5) where the integral is taken over $\partial\Omega \backslash B(P,\varepsilon)$. Then for almost every $P \in B(P_0,4\delta)$, $J_i f(P) = \lim_{\varepsilon>0} J_{i,\varepsilon}f(P)$ exists. Moreover the mapping $f \to J_i f$ is compact from $L_1^p(B(P_0,\delta))$ into $L^p(B(P_0,4\delta))$.*

Proof. We first prove boundedness for maximal operators of the following types.

$$I_{\alpha,\beta}^* f(P) = \sup_{\varepsilon>0}\left|\int_{|P-Q|>\varepsilon}(f(Q)-f(P))\left(\frac{\partial^4}{\partial n_P^\alpha \partial t_P^\beta}\right)^Q F(P-Q;\vec{n}_P)\langle\vec{n}_P,\vec{t}_Q\rangle ds(Q)\right|$$

and

$$\tilde{I}_{\alpha,\beta}^* f(P) = \sup_{\varepsilon>0}\left|\int_{|P-Q|>\varepsilon}(f(Q)-f(P))\left(\frac{\partial^4}{\partial n_P^\alpha \partial t_P^\beta}\right)^Q F(P-Q;\vec{n}_P)\langle\vec{t}_P,\vec{t}_Q\rangle ds(Q)\right|$$

where $\alpha + \beta = 4$. From (2.3.5) it follows that

$$\text{(3.3.8)} \quad \left(\frac{\partial^4}{\partial n_P^\alpha \partial t_P^\beta}\right)^Q F(P-Q;n_P)$$
$$= \operatorname{Re}\left\{c_{\alpha,\beta}\int_\gamma \frac{(-\zeta)^\alpha d\zeta}{(\zeta^2+1)^2(\langle P-Q,\vec{n}_P\rangle\zeta + \langle P-Q,\vec{t}_P\rangle)^2}\right\}$$

where $c_{\alpha,\beta}$ is a real constant depending only upon α and β. Applying Cauchy's integral formula in (3.3.8) it follows that

$$\text{(3.3.9)} \quad \left(\frac{\partial^4}{\partial n_P^\alpha \partial t_P^\alpha}\right)^Q F(P-Q;n_P) = \sum_{\mu+\nu=4} c_{\mu,\nu}\frac{\langle P-Q,\vec{n}_P\rangle^\mu\langle P-Q,\vec{t}_P\rangle^\nu}{|P-Q|^6}.$$

Thus for operators of the type $I_{\alpha,\beta}^* f$ it suffices to consider operators of the type

$$I_{\mu,\nu}^* f(P) = \sup_{\varepsilon>0}\left|\int_{|P-Q|>\varepsilon}\frac{(f(P)-f(Q))\langle P-Q,\vec{n}_P\rangle^\mu\langle P-Q,\vec{t}_P\rangle^\nu}{|P-Q|^6}\langle\vec{n}_P,\vec{t}_Q\rangle ds(Q)\right|.$$

Letting $P = (x,\phi(x))$ and $Q = (z,\phi(z))$ in local coordinates with origin at P_0, we are led to analyze Euclidean operators of the form

$$\text{(3.3.10)} \quad \sup_{\varepsilon>0}\left|\int_{|x-z|^2+(\phi(x)-\phi(z))^2>\varepsilon^2}\frac{(f(x)-f(z))(\phi'(x)-\phi'(z))}{((x-z)^2+(\phi(x)-\phi(z))^2)^3}\right.$$
$$\cdot\left(\frac{-\phi'(x)(x-z)+(\phi(x)-\phi(z))}{\sqrt{1+\phi'(x)^2}}\right)^\mu\left(\frac{(x-z)+\phi'(x)(\phi(x)-\phi(z))}{\sqrt{1+\phi'(x)^2}}\right)^\nu$$
$$\left.\cdot\frac{1}{\sqrt{1+\phi'(x)^2}}dz\right|$$

from $L_1^p(\mathbf{R})$ into $L^p(\mathbf{R})$ where $\mu + \nu = 4$. Expanding (3.3.10) our problem finally reduces to analyzing Euclidean operators of the form

$$(3.3.11)\quad \sup_{\varepsilon>0}\left|\int_{|x-z|^2+(\phi(x)-\phi(x))^2>\varepsilon^2} (f(x)-f(z))(\phi'(x)-\phi'(z)) \cdot \frac{(x-z)^\sigma(\phi(x)-\phi(z))^\tau}{((x-z)^2+(\phi(x)-\phi(z))^2)^3(\sqrt{1+\phi'(x)^2})}\,dz\right|$$

where $\sigma + \tau = 4$. Let us denote the integral in (3.3.11) by $K^*_{\sigma,\tau}f(x)$ and let $L^*_{\sigma,\tau}f(x)$ denote the corresponding maximal operator in which the integration is taken over $|x - z| > \varepsilon$. Then it is not difficult to see that (see (3.2.6))

$$(3.3.12)\quad |K^*_{\sigma,\tau}f(x) - L^*_{\sigma,\tau}f(x)| \le c \sup_{\varepsilon>0} \varepsilon^{-1}\int_{|x-z|\le\varepsilon}\left|\frac{f(x)-f(z)}{x-z}\right| dz \le c\,Mf'(x)$$

where $Mf'(x)$ denotes the Hardy-Littlewood maximal function of f'. Breaking up $L^*_{\sigma,\tau}f(x)$ into two parts—one with $\phi'(x)$ and the other with $\phi'(z)$—it follows from Calderón's theorem [3], Theorem 4 that the operator $f \to L^*_{\sigma,\tau}f$ is bounded from $L_1^p(\mathbf{R})$ into $L^p(\mathbf{R})$. It now follows that the operator $f \to I^*_{\alpha,\beta}f$ is bounded from $L_1^p(B(P_0,\delta))$ into $L^p(B(P_0,4\delta))$. The proof for integrals of the type $\tilde{I}^*_{\alpha\beta}f$ is identical with the proof just given except that in (3.3.10) $\phi'(x) - \phi'(z)$ is replaced by $1 + \phi'(x)\phi'(z)$. Let $J_i^*f(P) = \sup_{\varepsilon>0}|J_{i,\varepsilon}f(P)|$. It now follows that each of the operators $f \to J_i^*f$ is bounded from $L_1^p(B(P_0,\delta))$ into $L^p(B(P_0,4\delta))$, $1 \le i \le 4$.

We now show that the integrals $J_if(P)$, $1 \le i \le 4$, exist as principal valued integrals for almost every $P \in B(P_0,4\delta) \cap \partial\Omega$ and that the operators $f \to J_if$ are compact from $L_1^p(B(P_0,\delta))$ into $L^p(B(P_0,4\delta))$. We first consider J_1f and J_2f. Let $k(x,z)$ denote the functions

$$A_1(\partial^Q_{t_P},\partial^Q_{n_P})\,\partial^P_{t_P}F(P-Q;\vec{n}_P) \quad\text{or}\quad A_1(\partial^Q_{t_P},\partial^Q_{n_P})\,\partial^P_{n_P}F(P-Q;\vec{n}_P)$$

where $P = (x,\phi(x))$ and $Q = (z,\phi(z))$ are in local coordinates with origin at P_0 and where $\phi \in C_0^1(\mathbf{R})$. J_1f and J_2f are then of the form

$$(3.3.13)\quad Tf(x) = \lim_{\varepsilon\to0} T_\varepsilon f(x) = \lim_{\varepsilon\to0}\int_{|x-z|>\varepsilon}(f(x)-f(z))k(x,z)\frac{\phi'(x)-\phi'(z)}{\sqrt{1+(\phi'(x))^2}}\,dz.$$

Let $\{\phi_j\}$ be a sequence in $C_0^\infty(\mathbf{R})$ with $\phi_j \to \phi$ and $\phi_j' \to \phi'$ uniformly. Let

$$(3.3.14)\quad T_jf(x) = \lim_{\varepsilon\to0} T_{j,\varepsilon}f(x) = \lim_{\varepsilon\to0}\int_{|x-z|>\varepsilon}(f(x)-f(z))k(x,z)\left(\frac{\phi_j'(x)-\phi_j'(z)}{\sqrt{1+(\phi'(x))^2}}\right)dz.$$

From simple estimates on the derivatives of the fundamental solution we have

$|k(x,z)| \leq c|x - z|^{-2}$. Since $|\phi_j'(x) - \phi_j'(z)| \leq c_j|x - z|$ it follows that the integral $T_jf(x)$ exists for almost every x, $j = 1, 2, \ldots$ and that each of the operators $f \rightarrow T_jf$ is compact from $L_1^p(I)$ into $L^p(I)$ where I is a fixed interval on $\mathbf{R}$. From the first part of the lemma, it follows that $(T_\varepsilon - T_{j,\varepsilon})f(x)$ can be written as a sum of integrals of the form

$$(3.3.15)\quad \int_{|x-z|>\varepsilon} \frac{(f(x) - f(z))[(\phi - \phi_j)'(x) - (\phi - \phi_j)'(z)](x - z)(\phi(x) - \phi(z))}{((x - z)^2 + (\phi(x) - \phi(z))^2)^3\sqrt{1 + \phi'(x)^2}}\, dz.$$

From the first part of the lemma, we have

$$(3.3.16)\quad \|\sup_{\varepsilon>0}(T_\varepsilon - T_{j,\varepsilon})f(\cdot)\|_{L^p(I)} \leq C\|f\|_{L_1^p(I)}(\|\phi - \phi_j\|_\infty + \|\phi' - \phi_j'\|_\infty).$$

It now follows from standard arguments that the integral in (3.3.13) exists for almost every x. Moreover, it also follows that $T_j \rightarrow T$ in norm and therefore the operator $f \rightarrow Tf$ is compact from $L_1^p(I)$ into $L^p(I)$. Finally, we note that the ε-truncated integrals for J_1f and J_2f differ from the ε-truncated integrals for Tf by a function which tends to 0 in L^p norm as $\varepsilon \rightarrow 0$ (see [4], Theorem 1.2). Thus $J_1f(x)$ and $J_2f(x)$ agree with the operators $Tf(x)$ almost everywhere and consequently the lemma is established for the operators J_1 and J_2.

We now consider the operators J_3 and J_4. Let $k(x,z)$ denote either of the functions

$$C_1(\partial_{tp}^Q,\partial_{np}^Q)\partial_{tp}^P F(P - Q;\vec{n}_P) \quad \text{or} \quad C_1(\partial_{tp}^Q,\partial_{np}^Q)\partial_{np}^P F(P - Q;\vec{n}_P)$$

in local coordinates with origin at P_0 with $P = (x,\phi(x))$ and $Q = (z,\phi(z))$, $\phi \in C_0^1(\mathbf{R})$. Let $\{\phi_j\}$ be a sequence in $C_0^\infty(\mathbf{R})$ with $\phi_j \rightarrow \phi$ and $\phi_j' \rightarrow \phi'$ uniformly. Let

$$k_j(x,z) = C_1(\partial_{tP_j}^{Q_j},\partial_{nP_j}^{Q_j})\partial_{tP_j}^{P_j}(F(P_j - Q_j;\vec{n}_P) \quad \text{or} \quad C_1(\partial_{tP_j}^{Q_j},\partial_{nP_j}^{Q_j})\partial_{nP_j}^{P_j}F(P_j - Q_j;\vec{n}_{P_j})$$

where $P_j = (x,\phi_j(x))$ and $Q_j = (z,\phi_j(z))$ in the same local coordinates. By Theorem 10.1 in [1] we have for $0 < \beta < 1$

$$(3.3.17)\qquad |k_j(x,z)| \leq c_{j,\beta}|x - z|^{-2+\beta}.$$

We note that this is a very delicate estimate and depends upon the role of C_1 as a conjugate Neumann operator. As before, we let

$$(3.3.18)\quad Tf(x) = \lim_{\varepsilon\rightarrow 0} T_\varepsilon f(x) = \lim_{\varepsilon\rightarrow 0}\int_{|x-z|>\varepsilon}(f(x) - f(z))k(x,z)\left(\frac{1 + \phi'(x)\phi'(z)}{\sqrt{1 + (\phi'(x))^2}}\right)dz$$

and

$$T_jf(x) = \lim_{\varepsilon\to 0} T_{j,\varepsilon}f(x) = \lim_{\varepsilon\to 0}\int_{|x-z|>\varepsilon} (f(x)-f(z))k_j(x,z)\left(\frac{1+\phi'(x)\phi'(z)}{\sqrt{1+(\phi'(x))^2}}\right)dz.$$

From (3.3.17), it follows that $T_jf(x)$ exists for almost every $x \in \mathbf{R}$ and that each of the operators $f \to T_jf$ is compact from $L_1^p(I)$ into $L^p(I)$. From the first part of the lemma it follows that

$$\text{(3.3.19)}\quad T_\varepsilon f(x) - T_{j,\varepsilon}f(x) = \sum_{\sigma+\tau=4} c(\sigma,\tau)\int_{|x-z|>\varepsilon} \frac{(f(x)-f(z))(1+\phi'(x)\phi'(z))(x-z)^\sigma}{\sqrt{1+(\phi'(x))^2}} \cdot\left\{\frac{(\phi(x)-\phi(z))^\tau}{((x-z)^2+(\phi(x)-\phi(z))^2)^3} - \frac{(\phi_j(x)-\phi_j(z))^\tau}{((x-z)^2+(\phi_j(x)-\phi_j(z))^2)^3}\right\}dz.$$

Let $\Psi_\tau(\zeta)$ denote the analytic function $\zeta^\tau/(1+\zeta^2)^3$ in a neighborhood of 0. Breaking up the term $(1+\phi'(x)\phi'(z))$ it follows from (3.3.19) that $T_\varepsilon f - T_{j,\varepsilon}f$ can be written as a sum of integrals of the form

$$\text{(3.3.20)}\quad h(x)\int_{|x-z|>\varepsilon}\left(\frac{g(z)}{x-z}\right)\left(\frac{f(x)-f(z)}{x-z}\right) \cdot\left\{\Psi_\tau\left(\frac{\phi(x)-\phi(z)}{x-z}\right) - \Psi_\tau\left(\frac{\phi_j(x)-\phi_j(z)}{x-z}\right)\right\}dz$$

where g and h have infinity norms bounded by a fixed constant. Now $\Psi_\tau(\zeta) - \Psi_\tau(\omega) = (\zeta-\omega)G_\tau(\zeta,\omega)$ where $G_\tau(\zeta,\omega)$ is analytic in a neighborhood of the origin in $\mathbf{C}^2$. Then the integral in (3.3.20) becomes

$$\text{(3.3.21)}\quad h(x)\int_{|x-z|>\varepsilon}\left(\frac{g(z)}{x-z}\right)\left(\frac{f(x)-f(z)}{x-z}\right)\left(\frac{(\phi-\phi_j)(x)-(\phi-\phi_j)(z))}{x-z}\right) G\left(\frac{\phi(x)-\phi(z)}{x-z},\frac{\phi_j(x)-\phi_j(z)}{x-z}\right)dz.$$

Applying Calderón's theorem [3], Theorem 4 to each of these integrals we obtain

$$\text{(3.3.22)}\qquad \|\sup_{\varepsilon>0}(T_\varepsilon - T_{j,\varepsilon})f(\cdot)\|_{L^p(I)} \le C\|f\|_{L_1^p(I)}\|\phi' - \phi_j'\|_\infty.$$

It now follows that $Tf(x)$ exists for almost every $x \in I$ and that T is the norm limit of T_j as $j \to \infty$. Thus T is compact from $L_1^p(I)$ into $L^p(I)$. Just as before, the results of the lemma now follow for the operators J_3 and J_4. This completes the proof of the lemma.

Remark. We note that the crucial estimate (3.3.17) can be extended to the case $\beta = 1$—i.e. if $\phi_j \in C^2$, one has $|k_j(x,z)| \le c|x-z|^{-1}$. This follows from noting that in (10.21) of [1] one in fact has $|(Q - Q_0)\cdot\vec{n}(Q_0)| \le \text{const}|Q - Q_0|^2$ if ϕ_j is C^2.

Returning to (3.3.5), if we break up each of the operators J_1 through J_4 just as in (3.2.39) it now follows from the preceding lemma that each of these operators is compact from $L_1^p(B(P_0,\delta))$ into $L^p(\partial\Omega)$. We now analyze the operators J_5 through J_{10}. We have

Lemma (3.3.23). *Let $f \in L^p(\partial\Omega)$ with support in $B(P_0,\delta)$ and for $P \in B(P_0,4\delta) \cap \partial\Omega$, let $J_i f(P)$ be as defined in (3.3.6), $5 \le i \le 10$. Then for almost every $P \in B(P_0,4\delta)$, $J_i f(P) = \lim_{\varepsilon\to 0} J_{i,\varepsilon}(P)$ exists. Moreover the mapping $f \to J_i f$ is compact from $L^p(B(P_0,\delta))$ into $L^p(B(P_0,4\delta))$.*

Proof. The proof of this lemma is very similar to the proof of the previous lemma and we will be brief. We first analyze the maximal operators

$$(3.3.24)\quad I^*_{\alpha,\beta}f(P) = \sup_{\varepsilon>0}\left|\int_{|P-Q|>\varepsilon} f(Q)\left(\frac{\partial^3}{\partial n_P^\alpha \partial t_P^\beta}\right)^Q F(P-Q;\vec{n}_P)\langle\vec{n}_P,\vec{t}_Q\rangle ds(Q)\right|$$

and

$$\tilde{I}^*_{\alpha,\beta}f(P) = \sup_{\varepsilon>0}\left|\int_{|P-Q|>\varepsilon} f(Q)\left(\frac{\partial^3}{\partial n_P^\alpha \partial t_P^\beta}\right)^Q F(P-Q;\vec{n}_P)\langle\vec{t}_P,\vec{t}_Q\rangle ds(Q)\right|$$

where $\alpha + \beta = 3$. Applying Cauchy's theorem to calculate the kernels, the problem is reduced to analyzing operators of the type

$$(3.3.25)\quad I^*_{\mu,\nu}f(P) = \sup_{\varepsilon>0}\left|\int_{|P-Q|>\varepsilon} f(Q)\frac{\langle P-Q,\vec{n}_P\rangle^\mu\langle P-Q,\vec{t}_P\rangle^\nu}{|P-Q|^4}\langle n_P,t_Q\rangle ds(Q)\right|$$

and

$$\tilde{I}^*_{\mu,\nu}f(P) = \sup_{\varepsilon>0}\left|\int_{|P-Q|>\varepsilon} f(Q)\frac{\langle P-Q,\vec{n}_P\rangle^\mu\langle P-Q,\vec{t}_P\rangle^\nu}{|P-Q|^4}\langle\vec{t}_P,\vec{t}_Q\rangle ds(Q)\right|$$

where $\mu + \nu = 3$. Writing out these integrals in local coordinates, we are led to analyze the Euclidean operators

$$(3.3.26)\quad \sup_{\varepsilon>0}\left|\int_{|x-z|^2+(\phi(x)-\phi(z))^2>\varepsilon^2} f(z) \cdot\frac{(\phi'(x)-\phi'(z))(x-z)^\sigma(\phi(x)-\phi(z))^\tau}{((x-z)^2+(\phi(x)-\phi(z))^2)^2\sqrt{1+\phi'(x)^2}}dz\right|$$

and

$$\sup_{\varepsilon>0}\int_{|x-z|^2+(\phi(x)-\phi(z))^2>\varepsilon^2} f(z)\frac{(1+\phi'(x)\phi'(z))(x-z)^\sigma(\phi(x)-\phi(z))^\tau}{((x-z)^2+(\phi(x)-\phi(z))^2)^2\sqrt{1+(\phi'(x))^2}}dz\Bigg|$$

where $\sigma + \tau = 3$. A routine argument shows that the integrals in (3.3.26) and the corresponding ones where the integration is taken over $|x - z| > \varepsilon$ differ by a fixed multiple of the Hardy-Littlewood maximal function. Finally breaking up the terms $(\phi'(x) - \phi'(z))$ and $(1 + \phi'(x)\phi'(z))$ and applying Calderón's theorem, we obtain the boundedness on $L^p(\mathbf{R})$ of these maximal operators. It now follows that the operators $I^*_{\alpha,\beta}$ and $\tilde{I}^*_{\alpha,\beta}$ are bounded from $L^p(B(P_0,\delta))$ into $L^p(B(P_0,4\delta))$.

We now prove the pointwise existence and compactness of the operators $J_i f$. We first consider J_5 through J_8. Let $k(x,z)$ denote the kernel of any of these operators in local coordinates with origin at P_0, and consider

$$(3.3.27) \qquad Tf(x) = \int_{\mathbf{R}} f(z)k(x,z)\left(\frac{\phi'(x) - \phi'(z)}{\sqrt{1 + (\phi'(x))^2}}\right) dz$$

where $\phi \in C^1_0(\mathbf{R})$. Let $\phi_j \to \phi$ and $\phi'_j \to \phi'$ where each ϕ_j is in $C^\infty_0(\mathbf{R})$ and let

$$(3.3.28) \qquad T_j f(x) = \int_{\mathbf{R}} f(z)k(x,z)\left(\frac{\phi'_j(x) - \phi'_j(z)}{\sqrt{1 + (\phi'(x))^2}}\right) dz.$$

From trivial estimates on derivatives of the fundamental solution, we have $|k(x,z)| \le C|x - z|^{-1}$. Thus for each j, the integral in (3.3.28) converges absolutely and moreover the operator $f \to T_j f$ is compact from $L^p(I)$ into $L^p(I)$. Letting $T_\varepsilon f(x)$ and $T_{j,\varepsilon} f(x)$ denote the truncated integrals in (3.3.27) and (3.3.28) we obtain

$$(3.3.29) \quad \|\sup_{\varepsilon>0}|(T_\varepsilon - T_{j,\varepsilon})f(\cdot)|\|_{L^p(I)} \le C\|f\|_{L^p(I)}(\|\phi - \phi_j\|_{L^\infty(I)} + \|\phi' - \phi'_j\|_{L^\infty(I)}).$$

This follows by breaking up $(T_\varepsilon - T_{j,\varepsilon})(f)(x)$ as in the first part of this lemma and using the results there for the maximal operators. It now follows that each of the operators $J_i f$, $5 \le i \le 8$ exists for almost every P in $B(P_0,4\delta)$ and each of these operators is compact from $L^p(B(P_0,\delta))$ into $L^p(B(P_0,4\delta))$.

Finally, we consider the operators $J_9 f$ and $J_{10} f$. Again let $k(x,z)$ denote either of the kernels for these operators in local coordinates with origin at P_0. Specifically, let $P = (x,\phi(x))$ and $Q = (z,\phi(z))$ where $\phi \in C^1_0(\mathbf{R})$. Let $\phi_j \to \phi$ and $\phi'_j \to \phi'$ uniformly with $\phi_j \in C^\infty_0(\mathbf{R})$, $j = 1, 2, \ldots$. Let $P_j = (x,\phi_j(x))$ and $Q_j = (z,\phi_j(z))$ and let $k_j(x,z)$ denote either of the kernels for $J_9 f$ or $J_{10} f$ with P_j and Q_j replacing P and Q. Consider the operators

$$(3.3.30) \qquad Tf(x) = \int_{\mathbf{R}} f(x)k(x,z)\left(\frac{1 + \phi'(x)\phi'(z)}{\sqrt{1 + \phi'(x)^2}}\right) dz$$

and

$$T_j f(x) = \int_{\mathbf{R}} f(z)k_j(x,z)\left(\frac{1 + \phi'(x)\phi'(z)}{\sqrt{1 + \phi'(x)^2}}\right) dz.$$

Again by Theorem 10.1 in [1] we have for $0 < \beta < 1$,

$$(3.3.31) \qquad |k_j(x,z)| \le C_{j,\beta}|x - z|^{-1+\beta}.$$

It now follows that the integral defining $T_j f(x)$ exists for almost every x and moreover the operator T_j is compact from $L^p(I)$ into itself. Letting $T_\varepsilon f(x)$ and $T_{j,\varepsilon} f(x)$ denote the truncated integrals in (3.3.30) we obtain as before

$$\|\sup_{\varepsilon>0}|(T_\varepsilon - T_{j,\varepsilon})f(\cdot)|\|_{L^p(I)} \le c\|f\|_{L^p(I)}(\|\phi - \phi_j\|_{L^\infty(I)} + \|\phi' - \phi_j'\|_{L^\infty(I)}). \tag{3.3.32}$$

From this it follows that each of the operators $J_9 f(x)$ and $J_{10} f(x)$ exists for almost every $P \in B(P_0,4\delta)$ and each of these operators is compact from $L^p(B(P_0,\delta))$ into $L^p(B(P_0,4\delta))$. This completes the proof of the lemma.

Again, if we split each of the operators J_5 through J_{10} up as in (3.2.39) we obtain that each of these operators is compact from $L^p(B(P_0,\delta))$ into $L^p(\partial\Omega)$. We put the preceding results together in the following

Theorem (3.3.33). *Let $\dot f \in L_1^p(\partial\Omega) \times L^p(\partial\Omega) \times L^p(\partial\Omega)$, $1 < p < \infty$, and let $\partial_t T\dot f(P)$ and $\partial_n T\dot f(P)$ be defined by (3.3.3) and (3.3.4) respectively. Then each of these operators exists as a principal valued integral for almost every $P \in \partial\Omega$. Also,*

$$\|\partial_t T\dot f\|_{L^p(\partial\Omega)} \le c\|\dot f\|_{p,1} \tag{3.3.34}$$

and

$$\|\partial_n T\dot f\|_{L^p(\partial\Omega)} \le c\|\dot f\|_{p,1}.$$

In addition, we have that each of these operators is compact from $L_1^p(\partial\Omega) \times L^p(\partial\Omega) \times L^p(\partial\Omega)$ into $L^p(\partial\Omega)$.

Proof. We write $\dot f = \sum_{j=1}^n \psi_j \dot f$ where $\{\psi_j\}$ is a smooth partition of unity subordinate to an appropriate open covering of $\partial\Omega$ and apply the preceding lemmas.

4. The Multiple Layer Potential Near the Boundary

In this chapter we study the multiple layer potential $u(\dot f;X)$ near the boundary of Ω. We will show that the multiple layer potential and its first order derivatives have non-tangential limits at almost every point $P \in \partial\Omega$. We will see that these "boundary values" are closely related to the operators studied in Chapter 3.

4.1. Boundary values of the multiple layer potential. Let $\dot f \in \mathcal{B}_p(\partial\Omega)$, $1 < p < \infty$, and let $u(\dot f;X)$ denote the multiple layer potential of $\dot f$ at $X \in \Omega$. In this section we show that the multiple layer potential itself has non-tangential boundary values almost everywhere. We first recall some of the geometry of cones with vertices on $\partial\Omega$ from [4]. For $0 < \alpha_0 < 1$, there exists a constant $\delta = \delta(\alpha_0,\Omega)$ such that for each $P \in \partial\Omega$, the set

$$\Gamma_{\alpha_0}(P,\delta) = \{X \in \mathbf{R}^2 : 0 < |X - P| < \delta \quad \text{and} \quad \langle X - P, \vec n_P\rangle > \alpha_0|X - P|\} \subset \Omega.$$

Moreover, there exists a finite covering of $\partial\Omega$, $\{B(P_j,\delta_j)\}_{j=1}^n$, such that

$$B(P_j,4\delta_j) \cap \Omega = B(P_j,4\delta_j) \cap \left\{(x,y) : y > \phi_j(X), \phi_j \in C_0^1(\mathbf{R}), \|\phi_j'\|_\infty \le \frac{\alpha_0}{6}\right\}.$$

We assume in this section that $\delta \le \delta_j$ for $j = 1, \ldots, n$, our finite covering is fixed and $\|\phi_j'\|_\infty$ small enough to guarantee the estimates of Chapter 3. Let $\{\psi_j\}$ be a smooth partition of unity subordinate to $\{B(P_j,\delta_j)\}_{j=1}^n$. Let $T\dot{f}$ be defined on $\partial\Omega$ as in (3.2.41). The main result of this section is the following

Theorem (4.1.1). *Let $\dot{f} \in \mathcal{B}_p(\partial\Omega)$, $1 < p < \infty$, and let $u(X) = u(\dot{f};X)$ denote the multiple layer potential of $\dot{f}$ at $X \in \Omega$. Fix $0 < \alpha_0 < 1$, $\delta > 0$, $\{B(P_j,\delta_j)\}_{j=1}^n$ and $T\dot{f}$ as above. Then for almost every $P \in \partial\Omega$,*

$$u(\dot{f};X) \to f(P) + T\dot{f}(P) \tag{4.1.2}$$

as $X \to P$, $X \in \Omega$, $\langle X - P,\vec{n}_P\rangle > \alpha_0|X - P|$. Also for almost every $P \in \partial\Omega$, one has

$$u(\dot{f},X) \to -f(P) + T\dot{f}(P) \tag{4.1.3}$$

as $X \to P$, $X \in \mathbf{R}^2\backslash\bar{\Omega}$, $\langle X - P,\vec{n}_P\rangle < -\alpha_0|X - P|$.

Proof. We fix j and assume $\dot{f}$ has support in $B(P_j,\delta_j)$. We recall that $u(\dot{f};X)$ has the representation

$$u(\dot{f};X) = \int_{\partial\Omega} \dot{f}(Q)\mathcal{R}(P_j)^T K(\partial^Q_{tP_j},\partial^Q_{nP_j})^T F(X - Q)\,ds(Q) \tag{4.1.4}$$

where

$$K(\partial^Q_{tP_j},\partial^Q_{nP_j}) = \langle \vec{n}_{P_j},\vec{t}_Q\rangle A(\partial^Q_{tP_j},\partial^Q_{nP_j}) + \langle \vec{t}_{P_j},\vec{t}_Q\rangle C(\partial^Q_{tP_j},\partial^Q_{nP_j}),$$

$A = (A_1,A_2,A_3)$ and $C = (C_1,C_2,C_3)$. We first analyze the "lower order" terms in $u(\dot{f};X)$—i.e. the ones involving A_i and C_i, $i = 2, 3$. Since these terms involve second order derivatives on $F(X - Q)$ we have for $i = 2, 3$

$$|A_i(\partial^Q_{tP_j},\partial^Q_{nP_j})F(X - Q)| \le c\log|X - Q| \tag{4.1.5}$$

and

$$|C_i(\partial^Q_{tP_j},\partial^Q_{nP_j})F(X - Q)| \le c\log|X - Q|.$$

Thus it follows that the lower order terms of the multiple layer potential extend continuously up to the boundary. If we write $u(\dot{f};X)$ as $u_1(\dot{f};X) + u_2(\dot{f};X)$ where $u_2(X)$ denotes that part of $u(X)$ involving the lower order terms, we have

$$\begin{aligned} u_2(\dot{f};X) \to \int_{\partial\Omega} &(t_1(P_j)g(Q) + t_2(P_j)h(Q))\{\langle \vec{n}_{P_j},\vec{t}_Q\rangle A_2(\partial^Q_{tP_j},\partial^Q_{nP_j}) \\ &+ \langle \vec{t}_{P_j},\vec{t}_Q\rangle C_2(\partial^Q_{tP_j},\partial^Q_{nP_j})\}F(P - Q)\,ds(Q) \\ + \int_{\partial\Omega} &(t_1(P_j)h(Q) - t_2(P_j)g(Q))\{\langle \vec{n}_{P_j},\vec{t}_Q\rangle A_3(\partial^Q_{tP_j},\partial^Q_{nP_j}) \\ &+ \langle \vec{t}_{P_j},\vec{t}_Q\rangle C_3(\partial^Q_{tP_j},\partial^Q_{nP_j})\}F(P - Q)\,ds(Q) \end{aligned} \tag{4.1.6}$$

as $X \to P \in \partial\Omega$. We now analyze the principal part of the multiple layer potential $u_1(\dot{f};X)$. We introduce the non-tangential maximal function

$$(4.1.7) \qquad u_1^*(\dot{f};P) = \sup\{|u_1(\dot{f};X)| : X \in \Gamma_{\alpha_0}(P,\delta)\}.$$

We now show that $\|u_1^*\|_{L^p(\partial\Omega)} \le C_p\|\dot{f}\|_{p,1}$. To do this we use local coordinates with origin at P_j. In these coordinates, let $X = (x,t) \in \Omega$, $P = (x_0,\phi(x_0))$, and $Q = (z,\phi(z)) \in \partial\Omega$. In these coordinates, we have

$$(4.1.8) \quad u_1^*(P) = u_1^*(x_0) = \sup\{|u_1(x,t)| : t > \phi(x) \text{ and } t - \phi(x_0) - \phi'(x_0)(x - x_0) > \alpha_0\sqrt{1 + \phi'(x_0)^2}\,\sqrt{(x - x_0)^2 + (\phi(x) - \phi(x_0))^2}\}.$$

Suppose $u_1(x,t) = \int_{\partial\Omega} f(z)k(x,t,z)\,dz$. Then the proof of Theorem 1.3 in [4] shows that the mapping $f \to u_1^*$ is bounded from $L^p(B(P_j,\delta_j))$ into $L^p(\partial\Omega)$ provided the kernel $k(x,t,z) = k(X,Q)$ satisfies the following conditions

(4.1.9) i) $|k(X,Q)| \le C|X - Q|^{-1}$

ii) $|k(x,t,z)| \le c \min\left(\dfrac{1}{3|x - x_0|}, \dfrac{1}{t - \phi(x_0)}\right)$

whenever $|x_0 - z| \le \max(3|x - x_0|, t - \phi(x_0))$

iii) $|k(x,t,z) - k(x_0,t,z)| \le c|x - x_0|^{-1}$ whenever

$|x_0 - z| \ge \max(3|x - x_0|, t - \phi(x_0))$

iv) $|k(x_0,t,z) - k(x_0,\phi(x_0),z)| \le c|x - x_0|^{-1}$ whenever

$|x_0 - z| \ge \max(3|x - x_0|, t - \phi(x_0))$

v) the mapping $f \to \sup_{\varepsilon>0}\left|\displaystyle\int_{|z-x_0|>\varepsilon} f(z)k(x_0,\phi(x_0),z)\,dz\right|$ is bounded

on $L^p(\mathbf{R})$.

We now show that any kernel of the form

$$(4.1.10) \qquad k(x,t,z) = \frac{(x - z)^\alpha(t - \phi(z))^\beta}{(|x - z|^2 + (t - \phi(z))^2)^\gamma} \qquad 2\gamma = \alpha + \beta + 1$$

satisfies the five properties of (4.1.9). Property i) is immediate since $|x - z| \le |X - Q|$ and $|t - \phi(z)| \le |X - Q|$ while $|x - z|^2 + (t - \phi(z))^2 = |X - Q|^2$. To prove property iii), we write

(4.1.11) $k(x,t,z) - k(x_0,t,z)$

$$= \frac{((x-z)^\alpha - (x_0-z)^\alpha)(t-\phi(z))^\beta}{(|x-z|^2 + (t-\phi(z))^2)^\gamma} + (x_0 - z)^\alpha (t-\phi(z))^\beta \cdot \left\{ \frac{1}{(|x-z|^2 + (t-\phi(z))^2)^\gamma} - \frac{1}{(|x_0-z|^2 + (t-\phi(z))^2)^\gamma} \right\}$$
$$= I + II.$$

Standard estimates show

(4.1.12) $$|I| \leq \frac{c|x-z|^{\alpha-1}|x-x_0|}{(|x-z|^2 + (t-\phi(z))^2)^{\gamma-\beta/2}} \leq \frac{c'}{|x-x_0|}.$$

To estimate II we note that $|z - x_0| \geq 3|x - x_0|$ implies

$$|(|x-z|^2 + (t-\phi(z))^2)^{-\gamma} - (|x_0-z|^2 + (t-\phi(z))^2)^{-\gamma}| \leq \frac{c|x-x_0|}{|z-x_0|^{2\gamma+1}}$$

and $|t - \phi(z)| \leq t - \phi(x_0) + (\alpha_0/6)|z - x_0| \leq c|z - x_0|$. We have

(4.1.13) $$|II| \leq c\frac{|z-x_0|^{\alpha+\beta}|x-x_0|}{|z-x_0|^{2\gamma+1}} \leq \frac{c'}{|x-x_0|}.$$

This establishes property iii). To prove property iv) we write

(4.1.14) $k(x_0,t,z) - k(x_0,\phi(x_0),z)$

$$= \frac{(x_0-z)^\alpha((t-\phi(z))^\beta - (\phi(x_0)-\phi(z))^\beta)}{(|x_0-z|^2 + (t-\phi(z))^2)^\gamma} + (x_0-z)^\alpha(\phi(x_0)-\phi(z))^\beta \cdot \left\{ \frac{1}{(|x_0-z|^2 + (t-\phi(z))^2)^\gamma} - \frac{1}{(|x_0-z|^2 + (\phi(x_0)-\phi(z))^2)^\gamma} \right\}$$
$$= I + II.$$

Routine estimates show

(4.1.15) $$|I| \leq \frac{c|z-x_0|^{\alpha+\beta-1}|x-x_0|}{|z-x_0|^{2\gamma}} \leq \frac{c'}{|x-x_0|}$$

and

(4.1.16) $$|II| \leq \frac{c}{|x_0-z|^{2\gamma-\alpha-\beta}} \leq \frac{c'}{|x_0-x|}.$$

This proves property iv). Property v) follows immediately from Calderón's theorem.

We now write the principal part of the multiple layer potential as

$$(4.1.17)\quad u_1(X) = \int_{\partial\Omega} f(Q)A_1(\partial^Q_{t_{P_j}},\partial^Q_{n_{P_j}})F(X-Q;\vec{n}_{P_j})\langle\vec{n}_{P_j},\vec{t}_Q\rangle ds(Q) + \int_{\partial\Omega} f(Q)C_1(\partial^Q_{t_{P_j}},\partial^Q_{n_{P_j}})F(X-Q;\vec{n}_{P_j})\langle\vec{t}_{P_j},\vec{t}_Q\rangle ds(Q).$$

We consider integrals of the form

$$(4.1.18)\quad \int_{\partial\Omega} f(Q)\left(\frac{\partial^3}{\partial n^{\mu}_{P_j}\partial t^{\nu}_{P_j}}\right)^Q F(X-Q;\vec{n}_{P_j})\langle\vec{n}_{P_j},\vec{t}_Q\rangle ds(Q)$$

and

$$\int_{\partial\Omega} f(Q)\left(\frac{\partial^3}{\partial n^{\mu}_{P_j}\partial t^{\nu}_{P_j}}\right)^Q F(X-Q;\vec{n}_{P_j})\langle\vec{t}_{P_j},\vec{t}_Q\rangle ds(Q)$$

where $\mu + \nu = 3$. From Cauchy's formula we have

$$(4.1.19)\quad \left(\frac{\partial^3}{\partial n^{\mu}_{P_j}\partial t^{\nu}_{P_j}}\right)^Q F(X-Q) = \sum_{\sigma+\tau=3} c_{\sigma\tau}\frac{\langle X-Q,\vec{n}_{P_j}\rangle^{\sigma}\langle X-Q,\vec{t}_{P_j}\rangle^{\tau}}{|X-Q|^4}.$$

In terms of local coordinates with origin at P_j, we have

$$(4.1.20)\quad \frac{\langle X-Q,\vec{n}_{P_j}\rangle^{\sigma}\langle X-Q,\vec{t}_{P_j}\rangle^{\tau}}{|X-Q|^4} = \frac{(x-z)^{\tau}(t-\phi(z))^{\sigma}}{(|x-z|^2+(t-\phi(z))^2)^2}$$

which is of the form in (4.1.10). It now follows that the non-tangential maximal function of the principal part of the multiple layer potential is bounded on $L^p(\partial\Omega)$.

To establish the existence of the non-tangential limits almost everywhere, we must first show that these limits exist everywhere when $f \in C^1(\partial\Omega)$. Fix $P \in \partial\Omega$. Since $u((1,0,0);X) \equiv 2$ for $X \in \partial\Omega$, we can write

$$(4.1.21)\quad 2f(P) = \int_{\partial\Omega} f(P)A_1(\partial^Q_{t_{P_j}}\partial^Q_{n_{P_j}})F(X-Q)\langle\vec{n}_{P_j},\vec{t}_Q\rangle ds(Q) + \int_{\partial\Omega} f(P)C_1(\partial^Q_{t_{P_j}},\partial^Q_{n_{P_j}})F(X-Q)\langle\vec{t}_{P_j},\vec{t}_Q\rangle ds(Q).$$

We then have

$$(4.1.22)\quad u_1(f;X) = \int_{\partial\Omega} (f(Q)-f(P))A_1(\partial^Q_{t_{P_j}},\partial^Q_{n_{P_j}})F(X-Q)\langle\vec{n}_{P_j},\vec{t}_Q\rangle ds(Q) + \int_{\partial\Omega} (f(Q)-f(P))C_1(\partial^Q_{t_{P_j}},\partial^Q_{n_{P_j}})F(X-Q)\langle\vec{t}_{P_j},\vec{t}_Q\rangle ds(Q) + 2f(P).$$

Since $f \in C^1(\partial\Omega)$, as $X \to P$ we have

$$(4.1.23)\quad u_1(\dot f;X) \to \int_{\partial\Omega} (f(Q) - f(P))A_1(\partial^Q_{t_{P_j}}, \partial^Q_{n_{P_j}})F(P - Q)\langle \vec n_{P_j}, \vec t_Q\rangle ds(Q)$$
$$+ \int_{\partial\Omega} (f(Q) - f(P))C_1(\partial^Q_{t_{P_j}}, \partial^Q_{n_{P_j}})F(P - Q)\langle \vec t_{P_j}, \vec t_Q\rangle ds(Q)$$
$$+ 2f(P).$$

From (3.2.7) it follows that this limit is

$$(4.1.24)\qquad f(P) + \int_{\partial\Omega} f(Q)A_1(\partial^Q_{t_{P_j}}\partial^Q_{n_{P_j}})F(P - Q)\langle \vec n_{P_j}, \vec t_Q\rangle ds(Q)$$
$$+ \int_{\partial\Omega} f(Q)C_1(\partial^Q_{t_{P_j}}, \partial^Q_{n_{P_j}})F(P - Q)\langle \vec t_{P_j}, \vec t_Q\rangle ds(Q).$$

This combined with the L^p boundedness of the maximal operator $u_1^*(\dot f,\cdot)$ shows that the principal part of the multiple layer potential has non-tangential limits given by (4.1.24) for almost every $P \in \partial\Omega$. Combining (4.1.6) with this result it follows that if $\dot f$ has support in $B(P_j,\delta_j)$, then for almost every $P \in \partial\Omega$, $u(\dot f,X) \to f(P) + S(\dot f,P;P_j)$ as $X \to P$ non-tangentially where $S(\dot f,P;P_j)$ is given by (3.2.2). Finally, for $\dot f \in \mathcal{B}_p(\partial\Omega)$, we write $\dot f = \sum_{j=1}^n \dot f_j$ where $\{\psi_j\}$ is a smooth partition of unity subordinate to $\{B(P_j,\delta_j)\}_{j=1}^n$, and $\dot f_j$ is defined in Theorem (3.2.4). One then has

$$(4.1.30)\qquad u(\dot f;X) = \sum_{j=1}^n u(\dot f_j;X) \to \sum_{j=1}^n \psi_j(P)f(P) + \sum_{j=1}^n S(\dot f_j,P;P_j)$$
$$= f(P) + T\dot f(P)$$

as $X \to P$, $X \in \Omega$, $\langle X - P, \vec n_P\rangle > \alpha_0|X - P|$. The result for $X \to P$ non-tangentially, $X \in \mathbf{R}^2\backslash\bar\Omega$ is identical to the one just given except that the term $2f(P)$ is absent from (4.1.22) since $u((f(P),0,0);X) \equiv 0$ for $X \in \mathbf{R}^2\backslash\bar\Omega$. This completes the proof of the theorem.

4.2. Derivatives of the multiple layer potential near the boundary. In this section we prove the existence of non-tangential limits almost everywhere on $\partial\Omega$ of the derivatives of $u(\dot f;X)$ where $\dot f \in \mathcal{B}_p(\partial\Omega)$. Our main result of this section is the following

Theorem (4.2.1). *Let $\dot f = (f,g,h) \in \mathcal{B}_p(\partial\Omega)$, $1 < p < \infty$, and let $u(X)$ denote the multiple layer potential of $\dot f$ at $X \notin \partial\Omega$. Then for almost every $P \in \partial\Omega$*

$$(4.2.2)\qquad \partial_{t_P}u(X) \to t_1(P)g(P) + t_2(P)h(P) + \partial_{t_P}T\dot f(P)$$

and

$$\partial_{n_P} u(X) \to -t_2(P)g(P) + t_1(P)h(P) + \partial_{n_P} T\dot{f}(P)$$

as $X \to P$, $X \in \Omega$, $\langle X - P, \vec{n}_P \rangle > \alpha_0 |X - P|$. *Also*

$$\partial_{t_P} u(X) \to -(t_1(P)g(P) + t_2(P)h(P)) + \partial_{t_P} T\dot{f}(P) \tag{4.2.3}$$

and

$$\partial_{n_P} u(X) \to -(-t_2(P)g(P) + t_1(P)h(P)) + \partial_{n_P} T\dot{f}(P)$$

as $X \to P$, $X \notin \bar{\Omega}$, $\langle X - P, \vec{n}_P \rangle < -\alpha_0 |X - P|$. *The operators* $\partial_{t_P} T$ *and* $\partial_{n_P} T$ *are those in (3.3.33).*

Proof. We first prove an appropriate maximal inequality for the gradient of the multiple layer potential. For $P \in \partial\Omega$, we define

$$\nabla u^*(P) = \sup\{|\nabla u(X)| : X \in \Gamma_{\alpha_0}(P,\delta)\}. \tag{4.2.4}$$

Again we write $u(X) = u_1(X) + u_2(X)$ where $u_1(X)$ is the principal part of the multiple layer potential, and we work with u_1^* and u_2^* separately. We first consider $u_2^*(P)$. We assume $\dot{f}$ has support in $B(P_j, \delta_j)$ and work in local coordinates with origin at P_j and the representation

$$\begin{aligned} u_2(X) = \int_{\partial\Omega} & (t_1(P_j)g(Q) + t_2(P_j)h(Q))\{\langle \vec{n}_{P_j}, \vec{t}_Q \rangle A_2(\partial^Q_{t_{P_j}}, \partial^Q_{n_{P_j}}) \\ & + \langle t_{P_j}, t_Q \rangle C_2(\partial^Q_{t_{P_j}}, \partial^Q_{n_{P_j}})\} F(X - Q)\, ds(Q) \\ + \int_{\partial\Omega} & (t_1(P_j)h(Q) - t_2(P_j)g(Q))\{\langle \vec{n}_{P_j}, \vec{t}_Q \rangle A_3(\partial^Q_{t_{P_j}}, \partial^Q_{n_{P_j}}) \\ & + \langle \vec{t}_{P_j}, \vec{t}_Q \rangle C_3(\partial^Q_{t_{P_1}}, \partial^Q_{n_{P_j}})\} F(X - Q)\, ds(Q). \end{aligned} \tag{4.2.5}$$

We represent $\nabla u_2(X)$ as $(\partial_{t_{P_j}} u_2(X), \partial_{n_{P_j}} u_2(X))$ and work with each component separately. It follows that each component of $\nabla u_2(X)$ is a finite sum of terms of the form

$$\int_{\partial\Omega} f(Q) \left(\frac{\partial^3}{\partial n^\alpha_{P_j} \partial t^\beta_{P_j}} \right)^Q F(X - Q; n_{P_j}) \langle \vec{n}_{P_j}, \vec{t}_Q \rangle\, ds(Q) \tag{4.2.6}$$

or

$$\int_{\partial\Omega} f(Q) \left(\frac{\partial^3}{\partial n^\alpha_{P_j} \partial t^\beta_{P_j}} \right)^Q F(X - Q; \vec{n}_{P_j}) \langle \vec{t}_{P_j}, \vec{t}_Q \rangle\, ds(Q).$$

These are exactly the types of integrals considered in (4.1.18). From the proof of Theorem (4.1.1) it follows that each of the operators

$$\partial_{t_{P_j}} u_2^*(P) = \sup\{|\partial_{t_{P_j}} u_2(X)| : X \in \Gamma_{\alpha_0}(P,\delta)\} \tag{4.2.7}$$

and

$$\partial_{n_{P_j}} u_2^*(P) = \sup\{|\partial_{n_{P_j}} u_2(X)| : X \in \Gamma_{\alpha_0}(P,\delta)\}$$

is bounded from $L^p(B(P_j,\delta))$ into $L^p(\partial\Omega)$. Consequently, the same is true for the operator u_2^*. We now consider the operator u_1^*. We still assume $\dot{f}$ has support in $B(P_j,\delta_j)$ and work in local coordinates with origin at P_j. By (2.5.8) we may write

$$(2.4.8)\quad u_1(X) = \int_{\partial\Omega} f(Q)\partial_{n_Q}\Delta F(X-Q)\,ds(Q) + 2\int_{\partial\Omega} f(Q)\,\partial_{t_Q}(\partial^Q_{n_{P_j}},\partial^Q_{t_{P_j}})F(X-Q)\,ds(Q) \equiv u_{11}(X) + 2u_{12}(X).$$

Now $u_{11}(X)$ is the classical double layer potential of f at X. If we define $\nabla u_{11}^*(P) = \sup|\nabla u_{11}(x)|$ where the sup is taken over $X \in \Gamma_{\alpha_0}(P,\delta)$ it follows from Theorem 1.7 of [4] that the operator $f \to \nabla u_{11}^*$ is bounded from $L_1^p(B(P_j,\delta_j))$ into $L^p(\partial\Omega)$. Thus we need only establish similar boundedness for the operator $\nabla u_{12}^*(P) = \sup|\nabla u_{12}(X)|$ where the sup is taken over $X \in \Gamma_{\alpha_0}(P,\delta)$. Replacing $F(X-Q)$ by $F(X-Q;\vec{n}_{P_j})$ and noting (3.2.9) we have

$$(4.2.9)\qquad u_{12}(X) = u_{12}(x,t) = \int_{\mathbf{R}} f(z)\frac{\partial}{\partial z}\left\{\frac{(x-z)(t-\phi(z))}{(x-z)^2+(t-\phi(z))^2}\right\}dz \equiv \int_{\mathbf{R}} f(z)k(x,t,z)\,dz.$$

Since $\int_{\mathbf{R}} k(x,t,z)\,dz = 0$, we can write

$$(4.2.10)\qquad u_{12}(x,t) = \int_{\mathbf{R}} (f(z) - f(x_0))k(x,t,z)\,dz.$$

Then

$$\nabla u_{12}(x,t) = \nabla_{x,t}u_{12}(x,t) = \int_{\mathbf{R}} f(z)\nabla_{x,t}k(x,t,z)\,dz$$

and we need only estimate the maximal function

$$(4.2.11)\qquad \nabla u_{12}^*(x_0) = \sup\left|\int_{\mathbf{R}} (f(z) - f(x_0))\nabla_{x,t}k(x,t,z)\,dz\right|$$

where the sup is taken over the set

$$\{(x,t): t > \phi(x),\ \text{ and }\ t - \phi(x_0) - \phi'(x_0)(x-x_0) > \alpha_0\sqrt{1+\phi'(x_0)^2}\sqrt{(x-x_0)^2+(\phi(x)-\phi(x_0))^2}\}.$$

A proof very similar to that in Theorem (4.1.1) shows that the operator $f \to \nabla u^*_{12}$ is bounded from $L^p_1(B(P_j,\delta_j))$ into $L^p(\partial\Omega)$ provided $\nabla_{x,t}k(x,t,z)$ satisfies the following properties:

(4.2.12) i) $|\nabla_{x,t}k(x,t,z)| = |\nabla_{x,t}k(X,Q)| \le c|X - Q|^{-2}$

ii) $|\nabla_{x,t}k(x,t,z)| \le c \min((3|x - x_0|)^{-2},(t - \phi(x_0))^{-2})$

whenever $|x_0 - z| \le \max(3|x - x_0|, t - \phi(x_0))$

iii) $|\nabla_{x,t}k(x,t,z) - \nabla_{x,t}k(x_0,\phi(x_0),z)| \le c|x - x_0|^{-2}$

whenever $|x_0 - z| \le \max(3|x - x_0|, t - \phi(x_0))$

iv) $f \to \sup_{\varepsilon \to 0} \left| \int_{|x_0 - z| > \varepsilon} (f(z) - f(x_0)) \nabla_{x,t}k(x,\phi(x_0),z)\,dz \right|$ is bounded

from $L^p_1(\mathbf{R})$ into $L^p(\mathbf{R})$.

A straightforward calculation shows that $\nabla_{x,t}k(x,t,z)$ consists of a finite sum of terms of the form

$$(4.2.13) \qquad c_{\alpha,\beta,\gamma}\left\{\frac{(x - z)^{\alpha}(t - \phi(z))^{\beta}\phi'(z)^{\gamma}}{((x - z)^2 + (t - \phi(z))^2)^3}\right\}$$

where $c_{\alpha,\beta,\gamma}$ is a constant and $\alpha + \beta = 4$. These terms are very similar to those encountered in Theorem (4.1.1) except for a different relation between the exponents. A completely similar argument shows that properties i) through iii) hold for these kernels. Property iv) follows, of course, from Calderón's theorem. It now follows that ∇u^*_{12} is bounded from $L^p_1(B(P_j,\delta_j))$ into $L^p(\partial\Omega)$. Consequently, the same is true for ∇u^*_1. For an arbitrary $\dot{f} \in \mathcal{B}_p(\partial\Omega)$ we write $\dot{f} = \sum_{j=1}^n \dot{f}_j$ where $\dot{f}_j$ is defined in Theorem (3.2.40) for a smooth partition of unity subordinate to $B(P_j,\delta_j)$. Applying the preceding results to each $\dot{f}_j$ and summing we find that ∇u^* is bounded from $\mathcal{B}_p(\partial\Omega)$ into $L^p(\partial\Omega)$.

We now show that for every $P \in \partial\Omega$, $\partial_{t_P}u(\dot{f};X)$ and $\partial_{n_P}u(\dot{f};X)$ have limits as $X \to P$, $X \in \Gamma_{\alpha_0}(P,\delta)$ when $\dot{f}$ is the restriction to $\partial\Omega$ of a function and its x and y derivatives where f is C^∞ in a neighborhood of $\partial\Omega$. We define

$$(4.2.14) \qquad \Psi(X) = \nabla f(P)\cdot(X - P).$$

Then $\Psi(X)$ is a polynomial of degree 1 and so by (2.5.7) we have for $X \in \Omega$

$$(4.2.15) \qquad 2\Psi(x) = \int_{\partial\Omega} \Psi(Q)\mathcal{R}(P)^T K(\partial^Q_{t_P},\partial^Q_{n_P})^T F(X - Q)\,ds(Q).$$

Differentiating (4.2.15) in the direction t_P we obtain

$$(4.2.16) \qquad 2\partial_{t_P}f(P) = -\int_{\partial\Omega} \Psi(Q)\mathcal{R}(P)^T K(\partial^Q_{t_P},\partial^Q_{n_P})^T \partial^Q_{t_P}F(X - Q)\,ds(Q).$$

Now if $\tilde{f}(P) = (f(P),0,0)$, we have

(4.2.17) $\partial_{t_P} u(X)$

$$= -\int_{\partial\Omega} (\dot{f}(Q) - \tilde{f}(P))\mathcal{R}(P)^T K(\partial^Q_{t_P},\partial^Q_{n_P})\,\partial^Q_{t_P} F(X - Q)\,ds(Q)$$

$$= -\int_{\partial\Omega} (\dot{f}(Q) - \tilde{f}(P) - \dot{\Psi}(Q))\mathcal{R}(P)^T K(\partial^Q_{t_P},\partial^Q_{n_P})\,\partial^Q_{t_P} F(X - Q)\,ds(Q)$$

$$+ 2\partial_{t_P} f(P).$$

Since the first component of $\dot{f}(Q) - \tilde{f}(P) - \dot{\Psi}(Q)$ is $O(|P - Q|^2)$ while the second and third components are $O(|P - Q|)$, we may let $X \to P$ in (4.2.17) and obtain

(4.2.18) $\lim_{X\to P} \partial_{t_P} u(X)$

$$= -\int_{\partial\Omega} (\dot{f}(Q) - \tilde{f}(P) - \dot{\Psi}(Q))\mathcal{R}(P)^T K(\partial^Q_{t_P},\partial^Q_{n_P})\,\partial^Q_{t_P} F(P - Q)\,ds(Q)$$

$$+ 2\partial_{t_P} f(P).$$

By Theorem 11.1 of [1] (in particular 11.11, page 225) we have

$$(4.2.19)\qquad -\int_{\partial\Omega} \dot{\Psi}(Q)R(P)^T K(\partial^Q_{t_P},\partial^Q_{n_P})\,\partial^Q_{t_P} F(P - Q)\,ds(Q) = \partial_{t_P}\Psi(P).$$

Combining (4.2.18) and (4.2.19) we obtain

$$(4.2.20)\qquad \lim_{X\to P} \partial_{t_P} u(X) = -\int_{\partial\Omega} (\dot{f}(Q) - \tilde{f}(P))\mathcal{R}(P)^T K(\partial^Q_{t_P},\partial^Q_{n_P})\,\partial^Q_{t_P} F(P - Q)\,ds(Q)$$

$$+ \partial_{t_P} f(P)$$

$$= \partial_{t_P} f(P) + \partial_{t_P} T\dot{f}(P)$$

where $\partial_{t_P} T$ is the operator defined in Theorem (3.3.33). By a completely similar argument one shows that

$$(4.2.21)\qquad \lim_{X\to P} \partial_{n_P} u(X) = \partial_{n_P} f(P) + \partial_{n_P} T\dot{f}(P).$$

Observing that $\partial_{t_P} f(P) = t_1(P)f_x(P) + t_2(P)f_y(P)$ and $\partial_{n_P} f(P) = -t_2(P)f_x(P) + t_1(P)f_y(P)$, the theorem now follows from (4.2.20) and (4.2.21) combined with our estimate for the non-tangential maximal function. This completes the proof of the theorem.

We conclude this section with a few remarks. First, the above proof shows that as $X \to P$ non-tangentially, $X \notin \bar{\Omega}$, one has

$$\partial_{t_P} u(X) \to -(t_1(P)g(P) + t_2(P)h(P)) + \partial_{t_P} T\dot{f}(P)$$

and

$$\partial_{n_P} u(X) \to -(-t_2(P)g(P) + t_1(P)h(P)) + \partial_{n_P} T\dot{f}(P).$$

Second, from the above theorem we can also compute $\lim_{X\to P} \dot{u}(f;X) = \lim_{X\to P} (u(X),\partial_x u(X),\partial_y u(X))$ as $X \to P$ non-tangentially, $X \in \Omega$. One obtains

$$\partial_x u(X) \to g(P) + t_1(P)\partial_{t_P} T\dot{f}(P) - t_2(P)\partial_{n_P} T\dot{f}(P) \tag{4.2.22}$$

and

$$\partial_y u(X) \to h(P) + t_2(P)\partial_{t_P} T\dot{f}(P) + t_1(P)\partial_{n_P} T\dot{f}(P)$$

for almost every $P \in \partial\Omega$ as $X \to P$ non-tangentially, $X \in \Omega$. Finally we summarize the results in the following

Theorem (4.2.23). *Let $\dot{f} \in \mathcal{B}_p(\partial\Omega)$, $1 < p < \infty$, and let $\dot{u}(\dot{f};X)$ denote the multiple layer potential of $\dot{f}$ at X along with its x and y derivatives. Let $\mathcal{K}\dot{f}$ be defined by*

$$\mathcal{K}\dot{f}(P) = (T\dot{f}(P),\partial_{t_P} T\dot{f}(P),\partial_{n_P} Tf(P))\mathcal{R}(P). \tag{4.2.24}$$

Then for almost every $P \in \partial\Omega$, $\dot{u}(\dot{f};X) \to \dot{f}(P) + \mathcal{K}\dot{f}(P) = (I + \mathcal{K})\dot{f}(P)$ as $X \to P$ non-tangentially, $X \in \Omega$. Also $\dot{u}(\dot{f};X) \to (-I + \mathcal{K})\dot{f}(P)$ as $X \to P$ non-tangentially, $X \notin \bar{\Omega}$.

5. Invertibility

In Chapter 2 we defined the multiple layer potential $u(\dot{f};X)$. In Chapters 3 and 4 we showed the vector valued function $\dot{u}(X) = (u,u_x,u_y)$ has interior non-tangential limits of the form $(I + \mathcal{K})\dot{f}$ where the boundary operator $\mathcal{K}$ is compact from $\mathcal{B}_p$ to $\mathcal{B}_p$.

Since $\Delta^2 u(X) = 0$ for $X \notin \partial\Omega$, we can solve the interior Dirichlet problem if $I + \mathcal{K}$ is invertible. The solution is given by the multiple layer potential $u((I + \mathcal{K})^{-1}\dot{f};X)$.

By the Fredholm theory it suffices to show $I + \mathcal{K}$ is one-to-one or equivalently that its adjoint $(I + \mathcal{K})^*$ is one-to-one. Agmon showed in [1] that for Hölder continuous boundary data defined on a boundary $\partial\Omega$ of class $C^{1+\beta}$ ($\beta > 1/2$) the kernel of $I + K$ is zero. For the L^p data and C^1 boundary considered in this paper Agmon's argument does not apply and we work with the adjoint $(I + \mathcal{K})^*$.

5.1. The dual space, the adjoint and the lower order potential. Recall from Section (2.4) that the dual space $\mathcal{B}_p^*$ is the quotient space

$$L^q(\partial\Omega) \times L^q(\partial\Omega) \times L^q(\partial\Omega)/\mathcal{B}_p^{\perp}$$

where $\mathcal{B}_p^{\perp}$ is the annihilator of $\mathcal{B}_p$. Recall from (2.4.2) that the symbol

$$\langle \dot{f},\dot{\theta}\rangle = \int_{\partial\Omega} f(Q)\theta(Q) + g(Q)\phi(Q) + h(Q)\psi(Q)\,ds(Q)$$

denotes the action of $\dot\theta \in \mathcal{B}_p^*$ on $\dot f \in \mathcal{B}_p$. With this notation $(I + K)^*$ is the operator on $\mathcal{B}_p^*$ satisfying

$$\langle \dot f, (I + \mathcal{K})^* \dot\theta \rangle = \langle (I + \mathcal{K}) \dot f, \dot\theta \rangle$$

for all $\dot f \in \mathcal{B}_p$ and $\dot\theta \in \mathcal{B}_p^*$.

In the solution of Laplace's equation for a C^1 domain the adjoint of the boundary values of the double layer potential are obtained as the normal derivative of the classical single layer potential. (See Theorem 1.10 of [4].) To find an analogous characterization of the adjoint of the boundary values of the multiple layer potential and its gradient recall (2.5.3). The integral representation is

$$\dot u(X) = \int_{\partial\Omega} \dot f(Q) \mathrm{k}(X,Q)\, ds(Q)$$

where $\mathrm{k}(X,Q) = K(\partial_x^Q, \partial_y^Q) D(\partial_x^X, \partial_y^X) F(X - Q)$ and $X \notin \partial\Omega$.

If the data f is Hölder continuous and the boundary $\partial\Omega$ is smoother than C^1, the operator $\mathcal{K}$ can be obtained by replacing X by $P \in \partial\Omega$ in the kernel matrix k. While it is clear from Chapters 3 and 4 that for the data and boundaries considered in this paper the representation of $\mathcal{K}$, (4.2.24) is more complicated, it is useful to proceed formally, assuming for the moment that

$$\mathcal{K}\dot f(P) = \int_{\partial\Omega} \dot f(Q) \mathrm{k}(P,Q)\, ds(Q).$$

The adjoint K^* acts on the dual space $\mathcal{B}_p^*$. Continuing formally, for $\dot\theta \in \mathcal{B}_p^*$

$$
\begin{aligned}
(5.1.1) \qquad \mathcal{K}^* \dot\theta(Q) &= \int_{\partial\Omega} \mathrm{k}(P,Q) \dot\theta(P)^T ds(P) \\
&= \int_{\partial\Omega} K(\partial_x^Q, \partial_y^Q)^T D(\partial_x^P, \partial_y^P) F(P - Q) \dot\theta(P)^T ds(P) \\
&= K(\partial_x^Q, \partial_y^Q)^T \int_{\partial\Omega} \theta(P) F(P - Q) + \phi(P) \partial_x^P F(P - Q) \\
&\qquad + \psi(P) \partial_y^P F(P - Q)\, ds(P).
\end{aligned}
$$

These calculations, while only formal, suggest the following analogue of the single layer potential.

Definition (5.1.2). The lower order potential $v = v(\dot\theta; X)$ with density $\dot\theta = (\theta, \phi, \psi) \in L^q \times L^q \times L^q$ is defined:

$$v(\dot\theta; X) = \int_{\partial\Omega} \theta(P) F(P - X) + \phi(P) \partial_x^P F(P - X) + \psi(P) \partial_y^P F(P - X)\, ds(P)$$

where F is the fundamental solution given in (2.3.1).

Theorem (5.1.3). *The lower order potential $v = v(\dot\theta; X)$ is well defined on* *the quotient space* $\mathcal{B}_p^*$ *and is globally* C^1. *Furthermore, if* $\dot{\theta} \in \operatorname{Ker}(I + \mathcal{H})^*$, $\langle \dot{f},\dot{\theta}\rangle = 0$ *for all polynomials* f *of degree less than or equal to one and* $v(\dot{\theta};X)$ *satisfies the estimates:*

$$v(X) = O(\log|x|) \qquad \text{as } |X| \to \infty \tag{5.1.4}$$

$$\frac{\partial^{\alpha+\beta} v}{\partial x^\alpha \partial y^\beta} = O(|X|^{-\alpha-\beta}) \qquad \text{as } |X| \to \infty \tag{5.1.5}$$

for $1 \le \alpha + \beta \le 3$.

Proof. Since $\mathcal{B}_p^*$ is a quotient space, we must show that different cosets define the same lower order potential. This follows because $D(\partial_x^P,\partial_y^P)F(P - X)$ is a compatible triple in the variable P. Then if $\dot{\theta} \in \mathcal{B}_p^\perp$,

$$v(\dot{\theta};X) = \langle D(\partial_x^P,\partial_y^P)F(P - X),\dot{\theta}\rangle = 0.$$

Next, it is clear that $v(X)$ is C^∞ for $X \notin \partial\Omega$. A routine calculation shows that v, v_x and v_y are continuous across the boundary $\partial\Omega$.

If $\dot{\theta} \in \operatorname{Ker}(I + \mathcal{H})^*$ and f is a polynomial of degree at most one then $(I + \mathcal{H})\dot{f} = 2\dot{f}$. So

$$\langle \dot{f},\dot{\theta}\rangle = \frac{1}{2}\langle (I + \mathcal{H})\dot{f},\dot{\theta}\rangle = \frac{1}{2}\langle \dot{f},(I + \mathcal{H})^*\dot{\theta}\rangle = 0.$$

Finally, for $\dot{\theta} \in \operatorname{Ker}(I + \mathcal{H})^*$, the lower order potential $v(X)$ with density $\dot{\theta}$ is defined by

$$\begin{aligned} v(X) &= \int_{\partial\Omega} \theta(P)F(P - X) + \phi(P)\partial_x^P F(P - X) + \psi(P)\partial_y^P F(P - X)\,ds(P) \\ &= \int \{\theta(P)(F(P - X) - F(-X) - \partial_x^X F(-X)x(P) - \partial_y^X F(-X)y(P)) \\ &\quad + \phi(P)(\partial_x^X F(P - X) - \partial_x^X F(-X)) \\ &\quad + \psi(P)(\partial_y^X F(P - X) - \partial_y^X F(-X))\}\,ds(P) \end{aligned} \tag{5.1.6}$$

where $(x(P),y(P))$ denotes the (x,y)-coordinates of the point P. A routine calculation shows that for $\alpha + \beta = 3$,

$$\left|\left(\frac{\partial^3}{\partial x^\alpha \partial y^\beta}\right)^X (F(P - X) - F(-X) - \partial_x^X F(-X)x(P) - \partial_y^X F(-X)y(P))\right| = O(|X|^{-3}) \tag{5.1.7}$$

$$\left|\left(\frac{\partial^3}{\partial x^\alpha \partial y^\beta}\right)^X (\partial_x^X F(P - X) - \partial_x^X F(-X))\right| = O(|X|^{-3})$$

$$\left|\left(\frac{\partial^3}{\partial x^\alpha \partial y^\beta}\right)^X (\partial_y^X F(P - X) - \partial_y^X F(-X))\right| = O(|X|^{-3})$$

as $|X| \to \infty$. If each third order operator in (5.1.7) is replaced by a second order operator the terms are $O(|X|^{-2})$ as $|X| \to \infty$. Similar estimates for the first derivatives and the functions F, F_x, and F_y imply the estimates in (5.1.4) and (5.1.5).

5.2. Invertibility of $(I + \mathcal{K})^*$. In this section we prove that $I + \mathcal{K}$ is invertible. The main result is:

Theorem (5.2.1). *If $\dot\theta \in \mathcal{B}_p^*$ and $(I + \mathcal{K})^*\dot\theta = 0$ then $\dot\theta \in \mathcal{B}_p^\perp$.*

We point out that by the Fredholm theory, Theorem (5.2.1) implies that $I + \mathcal{K}$ is invertible. The proof of the theorem is given in several lemmas.

Lemma (5.2.2). *Assume $\dot\theta \in \mathrm{Ker}(I + \mathcal{K})^*$. Let $v = v(\dot\theta;X)$ be the lower order potential with density $\dot\theta$. Then*

$$\int\int_{\bar\Omega^c} \mathrm{Re}\, M(v)\bar M(v)\, dxdt = \langle -(I + \mathcal{K})\dot v, \dot\theta\rangle = 0 \tag{5.2.3}$$

where $\bar\Omega^c$ denotes the complement of $\bar\Omega$.

Proof. The proof consists primarily of applying the Green's formula (2.2.2) to the integral in (5.2.3). Since the region $\bar\Omega^c$ is unbounded and the lower order potential v may have singularities at the boundary, the integration by parts requires the employment of a partition of unity.

Let $\{B(P_j,\delta_j)\}_{j=1}^n$ be a covering of $\partial\Omega$ and $\{\phi_j\}_{j=1}^n$ a system of local coordinates so that

$$B(P_j,4\delta_j) \cap \bar\Omega^c = B(P_j,4\delta_j) \cap \{(x,y): y > \phi_j(x), \phi_j \in C_0^1(\mathbf{R}),$$
$$\phi_j(0) = \phi_j'(0) = 0 \text{ and } \|\phi_j'\|_\infty < m_0\}.$$

The constant m_0 is chosen small enough to guarantee the estimates obtained in Chapters 3 and 4. Let $D_\rho = \bar\Omega^c \cap B(0,\rho)$ where $\rho > 0$ is chosen large enough so that $\Omega \subset B(0,\rho/2)$. Let δ_0 denote the distance from $\partial\Omega$ to $\mathbf{R}^2 \setminus \bigcup_{j=1}^n B(P_j,\delta_j)$. Then there is an open set $\mathcal{O}$ such that

$$\mathcal{O} \cup \left(\bigcup_{j=1}^n B(P_j,\delta_j) \right) \supset D_\rho,$$

$$\mathcal{O} \supset \{X \in D_\rho : d(\partial\Omega,X) > \delta_0/2\}$$

and

$$\mathcal{O} \cap \{X \in D_\rho : d(\partial\Omega,X) \le \delta_0/4\} = \varnothing.$$

Let $\{\psi_j\}_{j=1}^{n+1}$ be a smooth partition of unity subordinate to the cover $B(P_1,\delta_1)$, $\ldots$, $B(P_n,\delta_n)$, $\mathcal{O}$. For $j = 1, \ldots, n$ we define

$$D_j = \bar{\Omega}^c \cap B(P_j; 4\delta_j)$$
$$\gamma_j = \partial\Omega \cap B(P_j; 4\delta_j).$$

For $0 < t < \delta_0/8$ we define

$$D_{j,t} = B(P_j; 4\delta_j) \cap \{(x,y): y > \phi_j(x) + t, x \in \mathbf{R}\}$$
$$\gamma_{j,t} = B(P_j; 4\delta_j) \cap \{(x, \phi_j(x) + t); x \in \mathbf{R}\}.$$

Finally we define

$$D_{N+1} = D_{N+1,t} = \mathcal{O} \cap D_\rho.$$

Since any derivative of $\sum_{j=1}^{n+1} \psi_j$ is zero on D_ρ, a simple computation establishes that on D_ρ:

$$\text{(5.2.4)} \qquad \operatorname{Re} M(v)\bar{M}(v) = \sum_{j=1}^{n+1} \operatorname{Re} \psi_j M(v)\bar{M}(v) = \sum_{j=1}^{n+1} \operatorname{Re} M(\psi_j v)\bar{M}(v).$$

Hence,

$$\text{(5.2.5)} \qquad \sum_{j=1}^{n+1} \int\int_{D_{j,t}} \operatorname{Re} \psi_j M(v)\bar{M}(v)\,dxdy = \sum_{j=1}^{n+1} \int\int_{D_{j,t}} \operatorname{Re} M(\psi_j v)\bar{M}(v)\,dxdy.$$

Since $\operatorname{Re} M(v)\bar{M}(v) = (v_{xx} - v_{yy})^2 + (2v_{xy})^2 \geq 0$ and $\psi_j \geq 0$, the integrands on the left-hand side of (5.2.5) are non-negative. Applying the monotone convergence theorem as $t \to 0$

$$\text{(5.2.6)} \qquad \lim_{t\to 0} \sum_{j=1}^{n+1} \int\int_{D_{j,t}} \operatorname{Re} \psi_j M(v)\bar{M}(v)\,dxdy = \sum_{j=1}^{n+1} \int\int_{D_j} \operatorname{Re} \psi_j M(v)\bar{M}(v)\,dxdy$$
$$= \sum_{j=1}^{n+1} \int\int_{D_\rho} \operatorname{Re} \psi_j M(v)\bar{M}(v)\,dxdy$$
$$= \int\int_{D_\rho} \operatorname{Re} M(v)\bar{M}(v)\,dxdy.$$

We now work on the right-hand side of (5.2.5). For a fixed $j \neq n + 1$ we integrate by parts via (2.2.6) in the coordinates t_{P_j}, n_{P_j}, determined by the unit tangent $\vec{t}_{P_j} = t_1(P_j)\vec{i} + t_2(P_j)\vec{j}$ and the unit normal $\vec{n}_{P_j} = -t_2(P_j)\vec{i} + t_1(P_j)\vec{j}$. Since we are integrating over a region complementary to Ω, $\vec{t}_{P_j}$ points clockwise around $\partial\Omega$ and $\vec{n}_{P_j}$ points into the region $\bar{\Omega}^c$. Integrating by parts over $D_{j,t}$ and using the fact that $\Delta^2 v = 0$ off of $\partial\Omega$, (5.2.7) $$\int\int_{D_{j,t}} \operatorname{Re} M(\psi_j v)\bar{M}(v)\,dxdt$$

$$= \int_{\gamma_{j,t}} (\psi_j(v(Q),(\psi_j v)_{t_{P_j}}(Q),(\psi_j v)_{n_{P_j}}(Q))K(\partial^Q_{t_{P_j}},\partial^Q_{n_{P_j}})^T v(Q)\,ds(Q)$$

We define

$$D(\partial_{t_{P_j}},\partial_{n_{P_j}};\vec{n}_{P_j}) = D(t_1(P_j)\partial^Q_{t_{P_j}} - t_2(P_j)\partial^Q_{n_{P_j}}, t_2(P_j)\partial^Q_{t_{P_j}} + t_1(P_j)\partial^Q_{n_{P_j}})$$

where we recall that $D(\partial_x,\partial_y) = (I,\partial_x,\partial_y)$.

Expanding v in the (t_{P_j},n_{P_j}) coordinate system, (5.2.7) can be expressed as

(5.2.8) $$\int_{\gamma_{j,t}} (\psi_j v)^{\cdot}(Q)\mathcal{R}(P_j)^T K(\partial^Q_{t_{P_j}},\partial^Q_{n_{P_j}})^T \int_{\partial\Omega} D(\partial^P_{t_{P_j}},\partial^P_{n_{P_j}};n_{P_j}) F(P-Q)\dot{\theta}(P)^T ds(P)\,ds(Q)$$

$$= \int_{\gamma_{j,t}} (\psi_j v)^{\cdot}(Q)\mathcal{R}(P_j)^T \int_{\partial\Omega} K(\partial^Q_{t_{P_j}},\partial^Q_{n_{P_j}})D(\partial_{t_{P_j}},\partial_{n_{P_j}};\vec{n}_{P_j}) F(P-Q)\dot{\theta}(P)^T ds(P)\,ds(Q)$$

$$= \int_{\partial\Omega}\left\{\int_{\gamma_{j,t}} (\psi_j v)^{\cdot}(Q)\mathrm{k}(P,Q;\vec{n}_{P_j})\,ds(Q)\right\}\dot{\theta}(P)^T ds(P)$$

where $\mathrm{k}(P,Q;\vec{n}_{P_j}) = \mathcal{R}(P_j)^T K(\partial_{t_{P_j}},\partial_{n_{P_j}})^T D(\partial_{t_{P_j}},\partial_{n_{P_j}};\vec{n}_{P_j})F(P-Q)$. (Recall the definition of the matrix $\mathcal{R}(P_j)^T$ given in Section 2.5.) The second line follows from passing the differential operator K^T inside the inner integral and the third line comes from interchanging the order of integration. The last two steps are justified by the fact that the distance from $\gamma_{j,t}$ to $\partial\Omega$ is positive for all $t > 0$.

We next make the change of variables $Q = Q' + t\vec{n}_{P_j}$ in (5.2.8). Because the differential operators K^T and D are invariant under parallel translation and the support of $\psi_j v$ is contained in $B(P_j,\delta_j)$ we can rewrite the inner integral in the last line of (5.2.8) as

(5.2.9) $$I_{j,t}(\psi_j v)^{\cdot}(P) = \int_{\gamma_j} (\psi_j v)^{\cdot}(Q' + t\vec{n}_{P_j})\mathrm{k}(P - t\vec{n}_{P_j}, Q'; \vec{n}_{P_j})\,ds(Q).$$

Remark (5.2.10). Because the support of ψ_j is contained in $B(P_j,\delta_j)$, $I_{j,t}(\psi_j v)^{\cdot}(P)$ is the multiple layer potential of $(\psi_j v)^{\cdot}$ evaluated at the point $P - t\vec{n}_{P_j}$. The point $P - t\vec{n}_{P_j} \in \Omega$ for small t since $\vec{n}_{P_j}$ points out of Ω. It follows from the maximal estimates in Chapter 4 and the fact that the kernels $(\partial^Q_{xx} - \partial^Q_{yy})F(X-Q)$ and $(\partial^Q_{xy}F(X-Q))$ are bounded that for $\dot{f} \in \mathcal{B}_p$ and support $\dot{f} \subset B(P_j,4\delta_j)$

(5.2.11) $$\|I_{j,t}(\dot{f})\|_{p,1} \le C\|\dot{f}\|_{p,1}$$

independent of $0 < t \leq \delta_0/8$ and

$$\lim_{t\to 0} I_{j,t}(\dot{f}) = -(I + \mathcal{K})\dot{f}(P) \quad \text{a.e.} \tag{5.2.12}$$

and in the $\|\cdot\|_{p,1}$ norm. $I_{j,t}(\dot{f})$ is defined as in (5.2.9) with $\dot{f}$ replacing $(\psi_j v)^{\cdot}$. The minus sign in the right-hand side of (5.2.12) follows from the fact that the integral is taken clockwise.

Remark (5.2.13). If we let $g_t(Q') = (\psi_j v)(Q' + t\vec{n}_{P_j}) - (\psi_j v)(Q')$ then we see that $\dot{g}_t \in \mathcal{B}_p$, support of $g_t \subset B(P_j, 4\delta_j) \cap \partial\Omega$ and

$$\|\dot{g}_t\|_{p,1} \to 0 \qquad \text{as } t \to 0 \tag{5.2.14}$$

$$(\psi_j v)^{\cdot}(Q' + tn_{P_j}) = \dot{g}_t(Q') + (\psi_j v)^{\cdot}(Q'). \tag{5.2.15}$$

Using the notation of (5.2.9) and remarks (5.2.10) and (5.2.13) we can rewrite the last line of (5.2.8) as

$$\int_{\partial\Omega} \{I_{j,t}(\dot{g}_t)(P) + I_{j,t}(\psi_j v)^{\cdot}(P)\}\dot{\theta}(P)^T ds(P) = \langle I_{j,t}(\dot{g}_t),\dot{\theta}\rangle + \langle I_{j,t}(\psi_j v)^{\cdot},\dot{\theta}\rangle. \tag{5.2.16}$$

Recalling that (5.2.8) comes from (5.2.7) and letting $t \to 0$ we see from Remarks (5.2.10) and (5.2.13)

$$\begin{aligned}\lim_{t\to 0} \iint_{D_{j,t}} \operatorname{Re} M(\psi_j v)\bar{M}(v)\,dxdt &= \lim_{t\to 0} \langle I_{j,t}(\dot{g}_t),\dot{\theta}\rangle + \lim_{t\to 0} \langle I_{j,t}(\psi_j v)^{\cdot},\dot{\theta}\rangle \\ &= 0 + \langle -(I + \mathcal{K})(\psi_j v)^{\cdot},\dot{\theta}\rangle \\ &= -\langle (\psi_j v)^{\cdot},(I + \mathcal{K})^* \dot{\theta}\rangle \\ &= 0\end{aligned} \tag{5.2.17}$$

since $\dot{\theta} \in \operatorname{Ker}(I + \mathcal{K})^*$.

We finally must consider integration over $D_{n+1} = \mathcal{O} \cap D_\rho$. Since the region is bounded and there are no singularities to worry about on D_{n+1}, we can apply (2.2.2) directly and get

$$\begin{aligned}\iint_{D_{n+1}} \operatorname{Re} M(\psi_{n+1} v)\bar{M}(v)\,dxdy &= \int_{|Q|=\rho} (\psi_{n+1} v)^{\cdot}(Q)K(\partial_x,\partial_y)^T v(Q)\,ds(Q) \\ &= \mathcal{O}\left(\frac{1}{\rho 2}\right)\end{aligned} \tag{5.2.18}$$

as $\rho \to \infty$ by the estimates for v in (5.1.4).

If we let $t \to 0$ in (5.2.5) and use (5.2.6), (5.2.17) and (5.2.18) we get, letting $\rho \to \infty$

$$\iint_{\bar{\Omega}^c} \operatorname{Re} M(v)\bar{M}(v)\,dxdy = 0 \tag{5.2.19}$$

which completes the proof of the lemma.

Corollary (5.2.20). *If* $\dot\theta \in \operatorname{Ker}(I + \mathcal{H})^*$ *and* $v = v(\dot\theta;X)$ *is the lower order potential with density* $\dot\theta$, *then* $\operatorname{Re} M(v)\bar{M}(v) \equiv 0$ *in* $\bar\Omega^c$.

Proof. This follows from Lemma (5.2.2) and the fact that

$$\operatorname{Re} M(v)\bar{M}(v) = (v_{xx} - v_{yy})^2 + (2v_{xy})^2 \geq 0.$$

Lemma (5.2.21) (Agmon). *Let* $u(x,y)$ *be a solution of class* C^{2m-1} *in a domain* $\mathcal{D}$ *of the pair of equations:*

$$M_1(\partial_x,\partial_y)u(x,y) = 0$$

$$M_2(\partial_x,\partial_y)u(x,y) = 0$$

where $M_1(\xi,\zeta)$ *and* $M_2(\xi,\zeta)$ *are relatively prime homogeneous polynomials of degree* m *in* ξ, ζ. *Then* u *is a polynomial of degree at most* $2m - 2$.

Proof. This is Lemma 13.1 of Agmon [1].

Lemma (5.2.22). *If* $\dot\theta \in \operatorname{Ker}(I + \mathcal{H})^*$ *and* $v = v(\dot\theta;X)$ *is the lower order potential with density* $\dot\theta$, *then* v *is a constant on* $\bar\Omega^c$ *and both the interior and exterior normal derivatives are zero at the boundary* $\partial\Omega$.

Proof. By Corollary (5.2.20) $(v_{xx} - v_{yy})^2 + (2v_{xy})^2 = 0$ in $\bar\Omega^c$. This implies that $v_{xx} - v_{yy} = v_{xy} = 0$ in $\bar\Omega^c$. Since $M_1(\xi,\eta) = \xi^2 - \eta^2$ and $M_2(\xi,\eta) = \xi\eta$ are relatively prime homogeneous polynomials of degree two, we can conclude from Agmon's lemma that v is a polynomial of degree at most two. By the estimates in Theorem (5.1.3), $v(X) = O(\log|X|)$ as $|X| \to \infty$. Hence v is constant in $\bar\Omega^c$. Since v is globally C^1 we can conclude from the continuity of the derivatives of v that both its interior and exterior normal derivatives are zero at the boundary $\partial\Omega$.

Lemma (5.2.23). *If* $f \in C^2(\Omega) \cap C^1(\bar\Omega)$, $\dot\theta \in \operatorname{Ker}(I + \mathcal{H})^*$ *and* $v = v(\dot\theta;X)$ *is the lower order potential with density* $\dot\theta$, *then*

$$0 = \int\int_\Omega \operatorname{Re} M(f)\bar{M}(v)\,dxdy = \langle(-I + \mathcal{H})\dot f,\dot\theta\rangle. \tag{5.2.24}$$

Proof. The argument is similar to the proof of Lemma (5.2.2). We begin by constructing a partition of unity.

We choose a covering of $\partial\Omega$ by balls $\{B(P_j,\delta_j)\}_{j=1}^n$ and local coordinates $\{\phi_j\}_{j=1}^n$ where

$$B(P_j,4\delta_j) \cap \Omega = B(P_j,4\delta_j) \cap \{(x,y): y > \phi_j(x), \phi_j \in C_0^1(\mathbf{R}),$$
$$\phi_j(0) = \phi_j'(0) = 0, \|\phi_j'\|_\infty < m_0\}.$$

Letting δ_0 be the distance from $\partial\Omega$ to $\mathbf{R}^2 \setminus \bigcup_{j=1}^n B(P_j,\delta_j)$ we can find an open set $\mathcal{O}$

such that

$$\mathcal{O} \cup \left(\bigcup_{j=1}^{n} B_j(P_j,\delta_j)\right) \supset \Omega, \qquad \mathcal{O} \supset \{X \in \Omega : d(\partial\Omega,X) > \delta_0/2\}$$

and

$$\mathcal{O} \cap \{X \in \Omega : d(\partial\Omega,X) \leq \delta_0/4\} = \varnothing.$$

Let $\{\psi_j\}_{j=1}^{n+1}$ be a smooth partition of unity subordinate to the cover $B(P_1,\delta_1), \ldots, B(P_n,\delta_n)$, $\mathcal{O}$. (Support $\psi_{n+1} \subset \mathcal{O}$.) For $j = 1, \ldots, n$ we define:

$$D_j = B(P_j,4\delta_j) \cap \Omega$$

$$\gamma_j = B(P_j,4\delta_j) \cap \partial\Omega.$$

For $0 < t < \delta_0/8$ we define

$$D_{j,t} = B(P_j,4\delta_j) \cap \{(x,y) : y > \phi_j(x) + t, x \in \mathbf{R}\}$$

$$\gamma_{j,t} = B(P_j,4\delta_j) \cap \{(x,\phi_j(x) + t), x \in \mathbf{R}\}.$$

Finally we define $D_{n+1} = D_{n+1,t} = \mathcal{O}$.

We begin by showing that

$$\text{(5.2.25)} \qquad \lim_{t\to 0} \sum_{j=1}^{n+1} \int\int_{D_{j,t}} \operatorname{Re} M(\psi_j f)\bar{M}(v)\,dxdy = \langle(-I + \mathcal{H})\dot{f},\dot{\theta}\rangle.$$

Proceeding as in the proof of Lemma (5.2.2) we integrate by parts in (t_{P_j},n_{P_j})-coordinates to get, for $j = 1, \ldots, n$

$$\text{(5.2.26)} \qquad \int\int_{D_{j,t}} \operatorname{Re} M(\psi_j f)\bar{M}(v)\,dxdy$$

$$= \int_{\partial\Omega} \left\{\int_{\gamma_{j,t}} (\psi_j f)^{\cdot}(Q)\mathrm{k}(P,Q;\vec{n}_{P_j})\,ds(Q)\right\}\dot{\theta}(P)^T ds(P)$$

$$= \int_{\partial\Omega} \left\{\int_{\gamma_j} (\psi_j f)^{\cdot}(Q' + t\vec{n}_{P_j})\mathrm{k}(P - t\vec{n}_{P_j},Q';\vec{n}_{P_j})\,ds(Q')\right\}\dot{\theta}(P)^T ds(P).$$

The last line in (5.2.26) follows from the change of variables $Q = Q' + t\vec{n}_{P_j}$. We note that unlike the proof of Lemma (5.2.2), $\vec{t}_{P_j}$ points counterclockwise around $\partial\Omega$ and $\vec{n}_{P_j}$ points into Ω. Since $P \in \partial\Omega$ and $t > 0$, this implies that $P - t\vec{n}_{P_j} \in \bar{\Omega}^c$. If we now define

$$II_{j,t}(\dot{f})(P) = \int_{\gamma_j} \dot{f}(Q)\mathrm{k}(P - t\vec{n}_{P_j},Q;\vec{n}_{P_j})\,ds(Q)$$

with $\dot{f} \in \mathcal{B}_p$ supported in $B(P_j,4\delta_j) \cap \partial\Omega$ then we have the following analogs of Remarks (5.2.10) and (5.2.13)

$$\|II_{j,t}(f)\|_{p,1} \leq c\|f\|_{p,1} \tag{5.2.27}$$

$$\lim_{t\to 0} II_{j,t}(f)(P) = (-I + \mathcal{H})f(P) \quad \text{a.e.} \tag{5.2.28}$$

and in the $\|\cdot\|_{p,1}$ norm. (5.2.28) differs from (5.2.12) because in this case we are dealing with the exterior limit of u rather than the interior limit and because the integration is taken counterclockwise.

Setting $g_t(Q') = (\psi_j f)(Q' + t\vec{n}_{P_j}) - (\psi_j f)(Q')$, letting $t \to 0$ and noting that ψ_{n+1} is compactly supported in $\mathcal{O}$ we have

$$\begin{aligned} \lim_{t\to 0} \sum_{j=1}^{n+1} \int\int_{D_{j,t}} \operatorname{Re} M(\psi_j f)\bar{M}(v)\,dxdy &= \lim_{t\to 0} \sum_{j=1}^{n+1} \int_{\partial\Omega} \{II_{j,t}(\dot{g}_t)(P) + II_{j,t}(\psi_j \dot{f})(P)\}\dot{\theta}(P)^T ds(P) \\ &= \sum_{j=1}^{n+1} \int_{\partial\Omega} (-I + \mathcal{H})(\psi_j f)^{\cdot}(P)\dot{\theta}(P)^T ds(P) \\ &= \langle(-I + \mathcal{H})\dot{f};\dot{\theta}\rangle. \end{aligned} \tag{5.2.29}$$

If we set $f = v$ in (5.2.29), apply identity (5.2.4) and use monotone convergence we have:

$$\begin{aligned} \int\int_{\Omega} \operatorname{Re} M(v)\bar{M}(v)\,dxdy &= \lim_{t\to 0} \sum_{j=1}^{n+1} \int\int_{D_{j,t}} \psi_j \operatorname{Re} M(v)\bar{M}(v)\,dxdy \\ &= \lim_{t\to 0} \sum_{j=1}^{n+1} \int\int_{D_{j,t}} \operatorname{Re} M(\psi_j v)\bar{M}(v)\,dxdy \\ &= \langle(-I + \mathcal{H})\dot{v},\dot{\theta}\rangle \\ &= 0. \end{aligned} \tag{5.2.30}$$

The last step is valid because by Lemma (5.2.22) $\dot{v} = (c,0,0)$ where c is a constant and $(-I + \mathcal{H})(c,0,0) = (0,0,0)$.

We know that v is continuous in $\mathbf{R}^2$ and constant in Ω^c. From (5.2.30) and Agmon's lemma we know that v is a polynomial of degree at most two in Ω. Since v is constant on $\partial\Omega$ we must have v constant in Ω. Substituting a constant for v in (5.2.29) we have

$$0 = \langle(-I + \mathcal{H})\dot{f},\dot{\theta}\rangle = \langle\dot{f},(-I + \mathcal{H})^*\dot{\theta}\rangle.$$

This concludes the proof of Lemma (5.2.23).

Proof of Theorem (5.2.1). If $\dot{\theta} \in \operatorname{Ker}(I + \mathcal{H})^*$, then by definition,

$$\langle\dot{f},(I + \mathcal{H})^*\dot{\theta}\rangle = 0. \tag{5.2.31}$$

On the other hand we know from Lemma (5.2.23) that if $f \in C^2(\bar{\Omega})$, then

$$0 = \langle(-I + \mathcal{H})\dot{f},\dot{\theta}\rangle = \langle\dot{f},(-I + \mathcal{H})^*\dot{\theta}\rangle.$$

For an arbitrary $\dot{f} \in \mathcal{B}_p$ we approximate $\dot{f}$ by smooth compatible triples and we get

$$\langle\dot{f},(-I + \mathcal{H})^*\dot{\theta}\rangle = 0 \tag{5.2.32}$$

for all $\dot{f} \in \mathcal{B}_p$. Subtracting (5.2.31) from (5.2.30) we get $2\langle\dot{f},\dot{\theta}\rangle = 0$ for any $\dot{f} \in \mathcal{B}_p$. Thus $\dot{\theta} \in B_p^{\perp}$. In other words $(I + \mathcal{H})^*$ is one-to-one.

Conclusion. The Fredholm theory tells us that $(I + \mathcal{H})^{-1}$ exists and so $u = u((I + \mathcal{H})^{-1}\dot{f};X)$ is biharmonic and $\dot{u}(X) \to \dot{f}(P)$ non-tangentially a.e. as $X \to P$.

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